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Restricted and Unrestricted Dominance

Welfare, Inequality and Povery Orderings

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Jean-Yves Duclos

Paul Makdissi

UNIVERSITÉ DE
SHERBROOKE
Faculté des lettres et sciences humaines
Département d'économique

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Jean-Yves Duclos
Université Laval

Paul Makdissi
Université de Sherbrooke
Vrije Universiteit Amsterdam

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Abstract

This paper extends the previous literature on the normative links between the measurement of poverty, social welfare and inequality. We show how, when the range of possible poverty lines is unbounded above, a robust ranking of absolute poverty may be interpreted as a robust ranking of social welfare, and a robust ranking of relative poverty may be interpreted as a robust ranking of inequality for any order of stochastic dominance. This interpretation is also valid when the maximum poverty line is bounded and for certain orders of stochastic dominance, so long as social welfare and inequality judgements are "censored''. We also develop new criteria of restricted inequality dominance, and find that they warn against the use of some popular indices of relative poverty and censored inequality. Finally, we illustrate geometrically how the new criteria of restricted inequality dominance extends the ranking power of previously proposed dominance criteria.

Résumé

Cet article propose un prolongement des résultats existant dans la littérature sur les liens normatifs entre les mesures de pauvreté, de bien-être social et d'inégalité. Lorsqu'il n'y a pas de borne maximale sur les seuils de pauvreté admissibles, nous montrons comment il est possible d'interpréter, d'une part, un classement de pauvreté absolue comme un classement de bien-être social et, d'autre part un classement de pauvreté relative comme un classement d'inégalité. Cette interprétation est valide pour tous les ordres de dominance stochastique. Elle est aussi valide lorsqu'il existe une borne supérieure aux seuils de pauvreté admissibles pour certains ordres de dominance stochastique. Les jugements de bien-être social et d'inégalité sont alors "censurés". Nous développons aussi des nouveaux critères de dominance restreinte pour les jugements d'inégalité censurée et discutons de la cohérence de ce critère avec des indices de pauvreté et d'inégalité censurée fréquemment utilisés. Finalement, nous illustrons géométriquement comment ces nouveaux critères de dominance d'inégalité restreinte élargissent le pouvoir de classement des critères de dominance existant.

Keywords: Welfare, Inequality, Poverty, Stochastic Dominance
JEL numbers: I32, D63

1 Introduction

The comparison of distributions of well-being, and in particular the development of methodologies to establish the normative robustness of such comparisons, has been the focus of a large literature in the last three decades. Pioneering work addressed the measurement of inequality (Kolm (1969), Atkinson (1970), and Dasgupta *et. al.* (1973)), using the now widely used Lorenz curves. This was extended to the measurement of social welfare by Shorrocks (1983) and to poverty by Atkinson (1987), Foster and Shorrocks (1988a, b), and Jenkins and Lambert (1997, 1998a, b). Foster and Shorrocks (1988a, b, and c) further integrated this literature by establishing that poverty orderings for the FGT (Foster, Greer and Thorbecke (1984)) indices which are robust to wide variations in the poverty line are in fact equivalent to stochastic dominance orderings of social welfare or inequality of orders one to three. They also showed that, under some circumstances, comparisons of absolute poverty that are robust up to a finite upper bound for ranges of poverty lines are equivalent to welfare comparisons for distributions that are censored at the same upper bound for orders of dominance one to three.

This paper first generalizes these first-, second-, and third-order equivalencies to a broader class of poverty, inequality and social welfare indices. This is fairly straightforward given the link between indices of the FGT class and the functions used to test for poverty dominance of various orders. It then establishes the conditions under which these equivalencies may hold for any higher-order dominance. A critical element of these conditions is the choice of a maximum bound for welfare dominance tests as well as for the range of admissible poverty lines in poverty dominance tests. A popular choice for the upper bound of social welfare comparisons is the maximal level of well-being found in a sample. We show, however, that extending that upper bound to infinity increases the ranking power of social welfare dominance criteria. This also reconciles welfare and poverty dominance orderings at any order of stochastic dominance.

We also introduce the concept of censored inequality comparisons. In doing this, we develop links between these censored inequality comparisons and relative poverty comparisons (such as comparisons of the proportions of individuals below a certain proportion of the mean). This is particularly relevant in the light of the increasing popularity of relative poverty comparisons across developed countries, and the suspicion often expressed that these are really comparisons of inequality. This also leads to the derivation of restricted inequality dominance criteria. Such criteria are in fact often implicitly (though incompletely) applied in the empirical literature on the distribution of well-being. Ordering tools frequently used include ratios of quantiles over means, ratios of quantiles and interquantile differences normalised by the mean. We show, however, that the latter two tools are not consistent with inequality dominance orderings of any order, and can thus conflict with commonly accepted ethical criteria for comparing inequality in distributions of well-being. We complete the analysis by illustrating geometrically how the ordinal ranking power of inequality dominance criteria is increased by a focus on relative poverty and restricted inequality dominance.

The rest of the paper is organized as follows. Section 2 defines classes of poverty, inequality, and social welfare indices. Section 3 uses these classes to develop various criteria for stochastic dominance. Section 4 establishes links between poverty comparisons, on the one hand, and comparisons of social welfare and inequality, on the other. Section 5 develops criteria for censored welfare and inequality comparisons. Section 6 discusses the links between inequality and relative poverty dominance and some of the indices often used in the empirical literature on the distribution of well-being. Section 7 presents a geometric analysis of the increase in the ordinal ranking power provided by restricted inequality dominance criteria, compared to some of the existing criteria and alternative approaches. Section 8 concludes.

2 Classes of Poverty, Welfare, and Inequality indices

We wish to compare two distributions of well-being y (income, for short), each of which is represented by a distribution function drawn from the set $\Im$:

$$\Im := \{F : [0, a] \to [0, 1] \,|F \text{ is nondecreasing, continuous and onto}\},$$

where F is an income distribution function and a is some value equal to (or exceeding) the maximum possible income. We consider four types of comparisons for these distributions: comparisons of absolute poverty, relative poverty, social welfare, and inequality. To compare absolute poverty, the analyst must first choose a poverty line, z, such that individuals earning less than this amount are identified as poor. Subsequently, a poverty index,

P^A, must be selected to aggregate the different individuals' contributions to poverty. In this paper we focus on additive poverty indices, such that:

$$P_F^A(z) = \int_0^a p_A(y,z)\, dF(y)\,, \tag{1}$$

where:

$$\left.\begin{array}{lll} p_A(y,z) \geq 0, & \text{if} & y \leq z, \\ p_A(y,z) = 0, & \text{if} & y > z. \end{array}\right\} \tag{2}$$

The function $p_A(y,z)$ represents the contribution to total poverty made by an individual with income y. A popular example of additive poverty indices is the class of FGT (Foster, Greer and Thorbecke (1984)) indices, defined as:

$$FGT_F(\alpha, z) = \int_0^z (z-y)^\alpha\, dF(y)\,. \tag{3}$$

Other examples of additive indices are the Ckakravarty (1983) poverty indices and the Watts (1968) index, defined as

$$W = \int_0^z \log(y/z)\, dF(y) \tag{4}$$

and which can in turn be seen as a transformation of a particular case of the class of Clark, Hemming and Ulph's (1981) second class of poverty indices. We regroup these additive indices into classes Π_A^s, $s = 1, 2, \ldots,$ of absolute poverty indices. These classes are defined by:

$$\Pi_A^s := \left\{ P^A | p_A(y,z) \in \widehat{C}^s \quad \text{and} \quad (-1)^i \frac{\partial p_A(y,z)}{\partial y^i} \geq 0 \quad \forall i = 1, 2, \ldots, s \right\}, \tag{5}$$

where $\hat{C}^s$ represents the set of functions that are s times piecewise differentiable on $[0, \infty)$ [1]. We will return later on page 6 to the interpretation of the conditions on the derivatives in (5).

[1] Notice that if the $(s-1)$-th derivative of a function is piecewise differentiable, the $(s-1)^{th}$ derivative is necessarily continuous and the function itself and its first $(s-2)$ derivatives are continuous and differentiable everywhere. Admitting functions of which the $(s-1)$-th derivative is piecewise differentiable allows us to include into Π_A^s the FGT class of measures (for $\alpha = s$), which is the most widely used class of poverty indices in the literature. Note also that the continuity condition we impose is more restrictive than that in Zheng (1999), which only postulates continuity on the interval $[0, z)$. This difference between his and our assumptions has implications for the analysis developed in this paper. Specifically, we are able to consider dominance criteria for orders greater than two, even when there is significant uncertainty on the value of the lower bounds for the ranges of possible poverty lines. We shall return to this later in the paper.

To compare relative poverty, the analyst must first select a function, $\xi(F)$, that sets the poverty line as a function of the income distribution F. In this paper we fix this threshold as a proportion of the mean of the distribution[2]. Thus, we have:

$$\xi(F) = \lambda \mu_F, \tag{6}$$

where μ_F is the mean of the distribution F and $\lambda \in [0, \infty)$. We might at first presume that the analyst is only interested in $\lambda \in [0, a/\mu_F]$, but considering the range of all positive reals will prove useful for the subsequent analysis.

In order to aggregate the various incomes' contribution to relative poverty, we consider again additive indices, defined as:

$$P_F^R(\lambda) = \int_0^a p_R(y, \lambda\mu_F)\, dF(y), \tag{7}$$

where

$$\left.\begin{array}{ll} p_R(y, \lambda\mu_F) \geq 0, & \text{if} \quad y \leq \lambda\mu_F, \\ p_R(y, \lambda\mu_F) = 0, & \text{if} \quad y > \lambda\mu_F, \end{array}\right\} \tag{8}$$

and where

$$p_R(y, \lambda\mu_F) \text{ is homogeneous of degree zero in } y \text{ and } \mu_F. \tag{9}$$

The function $p_R(y, \lambda\mu_F)$ measures the contribution an individual with income y makes to total poverty when the income distribution function is F. We use the classes of relative poverty indices Π_R^s, $s = 1, 2, \ldots$, defined as:

$$\Pi_R^s := \left\{ P^R | p_R(y, \lambda\mu_F) \in \widehat{C}^s \quad \text{and} \quad (-1)^i \frac{\partial^i p_R(y, \lambda\mu_F)}{\partial y^i} \geq 0 \quad \forall i = 1, 2, \ldots, s \right\}. \tag{10}$$

Turning now to social welfare, we consider utilitarian social welfare functions U, such that:

$$U_F = \int_0^a u(y)\, dF(y). \tag{11}$$

We focus on social welfare indices that belong to the classes Ω^s, $s = 1, 2, \ldots$, defined as:

$$\Omega^s = \left\{ U | u(y) \in \hat{C}^s \quad \text{and} \quad (-1)^i \frac{d^i u(y)}{dy^i} \leq 0 \quad \forall i = 1, 2, \ldots, s \right\}. \tag{12}$$

[2] This is often done in the empirical literature. For example, the EUROMOD simulation package uses this definition for the poverty line for poverty comparisons across countries of the European Union (cf. Immervoll *et al.* (1999)).

As Blackorby and Donaldson (1978) demonstrate, each of these social welfare functions can be associated with a traditional Atkinson-Kolm-Sen index of inequality of the form:

$$I_F^U = 1 - \frac{y_{e,F}}{\mu_F}, \tag{13}$$

where the equally-distributed-equivalent income y_e is implicitly defined as:

$$u\left(y_{e,F}\right) = U_F = \int_0^a u\left(y\right) dF\left(y\right). \tag{14}$$

These indices are relative inequality indices[3] if and only if the function $u\left(y\right)$ is isoelastic. Given the subset of Ω^s for which $u\left(y\right)$ is isoelastic, it is possible to define corresponding classes of relative inequality indices Υ^s. We have:

$$\Upsilon^s := \left\{ I^U \mid U \in \Omega^s \text{ and } u \text{ is isoelastic} \right\}. \tag{15}$$

At this point it is useful to supply a normative interpretation of the different classes of indices. For the indices of absolute poverty and welfare, when $s = 1$ the indices must be such that poverty weakly decreases while welfare weakly increases when an individual's income increases. These indices are thus of the Pareto type in addition to being symmetric in income (they obey the anonymity axiom). For indices of relative poverty and inequality, the same properties apply for $s = 1$ if we hold mean income artificially exogeneous to that change. Note that first-order classes of inequality indices are generally not considered in the literature, since unrestricted dominance rankings of distributions over such classes are not possible (the dominance curves necessarily cross, as we illustrate below on page 11 in Section 3). It is, however, possible to obtain restricted first-order inequality dominance rankings of two distributions, as we discuss below in Section 5.

When $s = 2$ these indices must respect the Pigou-Dalton principle of transfers. This principle postulates that a mean-preserving transfer of income from a higher-income person to a lower-income person constitutes a social improvement. The Pigou-Dalton criterion has been framed alternatively as a strong and as a weak axiom for the study of poverty indices (see Donaldson and Weymark (1986) and Zheng (1999)). In the weak version, the axiom says that the poverty index must increase following a transfer from one individual to another, wealthier, individual, providing that both are initially below the

[3] An inequality index is relative if $I_F = I_G$ whenever $F\left(y\right) = G\left(\lambda y\right)$ for all y and for any given $\lambda > 0$.

poverty line and the transfer does not lift the wealthier person above this threshold. The strong axiom postulates that the index must increase even if this transfer pushes the higher-income recipient above the poverty line. Our assumption of continuity of the poverty indices in the neighbourhood of the poverty line necessarily implies the strong version of the axiom.

When $s = 3$ the indices must also be sensitive to favourable composite transfers. These transfers are such that a beneficial Pigou-Dalton transfer within the lower part of the distribution, accompanied by a reverse Pigou-Dalton transfer within the upper part of the distribution, will add to welfare and decrease poverty and inequality, provided that the variance of the distribution is not increased. Kolm (1976) was the first to introduce this condition into the inequality literature, and Kakwani (1980) subsequently adapted it to poverty measurement (see also Shorrocks and Foster (1987) for a complete caracterization of this transfer principle). As with the Pigou-Dalton principle of transfers, it can also be formulated as a strong or as a weak axiom[4]. Our continuity assumption, $\hat{C}^3$, necessarily implies the strong version of the axiom. It should be noted, however, that Zheng (1999), while agreeing that an assumption of continuity of poverty indices at the poverty line is reasonable, argues that the continuity of their successive derivatives is more difficult to justify from a normative perspective. However, as Zheng (1999) also points out, if we do not make this assumption, increases in the ordinal ranking power are significantly diminished as we move to dominance orders greater than two. In contrast to Zheng (1999), and in line with Ravallion (1994), we thus prefer here to identify classes of indices for which the ordinal ranking power increases with increases in orders of dominance, and not, as in Zheng (1999), to highlight the consequences of not making sufficiently strong assumptions on the continuity of successive derivatives.

For the interpretation of higher orders of dominance, we can use the generalised transfer principles of Fishburn and Willig (1984). For $s = 4$, for instance, consider a combination of composite transfers, the first one being favourable and occuring within the lower part of the distribution, and the second one being unfavourable and occuring within the higher part of the distribution. Because the favourable composite transfer occurs lower down in the income distribution, indices that are members of the $s = 4$ classes should respond favourably to this combination of composite transfers. Generalised

[4]There are, in fact, three different formulations of the composite transfer. The interested reader may refer to Zheng (1997) for an informative discussion of this.

higher-order transfer principles essentially postulate that, as s increases, the weight assigned to the effect of transfers occurring at the bottom of the distribution also increases. Blackorby and Donaldson (1978) describe these indices as becoming more Rawlsian. As shown in Davidson and Duclos (2000) for poverty indices, when $s \to \infty$ only the lowest income counts, though as we will see later in Section 4 this result does not generalise to indices of welfare and inequality.

3 Criteria for Stochastic Dominance

This section presents a series of criteria for stochastic dominance in comparisons of absolute poverty, relative poverty, welfare, and inequality. These criteria shall be used in the Section 4 to establish links between absolute and relative poverty comparisons, on the one hand, and comparisons of welfare and inequality, on the other.

Consider two income distribution functions, F and G, both drawn from the set $\Im$. In order to simplify the exposition, we define $D_F^1(y) = F(y)$ and $D_F^s(y) = \int_0^y D_F^{s-1}(u)\,du$ for all integers $s \geq 2$. These curves can be expressed as

$$D_F^{s+1}(z) = (s!)^{-1} FGT_F(s, z). \tag{16}$$

$D_G^s(y)$ is defined similarly. Consider, first, comparisons of absolute poverty. When poverty does not increase in the passage from distribution F to distribution G, we have that:

$$\Delta P_{FG}^A(z) = \int_0^a p_A(y,z)\,dG(y) - \int_0^a p_A(y,z)\,dF(y) \leq 0. \tag{17}$$

If there is a consensus that the poverty line z should not exceed some maximum, z^+, then it is possible to lay out a necessary and sufficient condition for absolute poverty dominance applicable to all orders of stochastic dominance (for ease of exposition, the proofs of all propositions appear in the appendix).

Proposition 1 *A necessary and sufficient condition for $\Delta P_{FG}^A(z) \leq 0$, for any poverty line $z \in [0, z^+]$ and any absolute poverty index $P^A \in \Pi_A^s$, is:*

$$D_F^s(y) - D_G^s(y) \geq 0 \quad \forall y \leq z^+. \tag{DSPA}$$

The preceding proposition is in fact a generalization of Atkinson's (1987) results, which cover the cases of $s = 1, 2$. Condition DSPA is not entirely new, since a similar version is found in Foster and Shorrocks (1988a). Their analysis, however, deals only with the FGT class of indices, and not, as we do, with the entire Π_A^s classes of indices.

Consider now comparisons of relative poverty. If there is no increase in poverty subsequent to the passage from distribution F to distribution G, we have that:

$$\Delta P_{FG}^R(\lambda) = \int_0^a p_R(y, \lambda\mu_G)\, dG(y) - \int_0^a p_R(y, \lambda\mu_F)\, dF(y) \leq 0. \tag{18}$$

Consider relative poverty lines up to some maximum proportion of the mean, λ^+. To establish relative poverty dominance criteria, we first define transformations of the distribution functions F and G, $\widehat{F}$ and $\widehat{G}$, obtained by dividing each income in the original distribution by the average income in that distribution. The transformed incomes are thus $\widehat{y}_F = y/\mu_F$ for $\widehat{F}$ and $\widehat{y}_G = y/\mu_G$ for $\widehat{G}$. This yields

$$D_{\widehat{F}}^{s+1}(\lambda) = s!^{-1} \int_0^{\lambda\mu_F} \left(\frac{\lambda\mu_F - y}{\lambda\mu_F}\right)^{s-1} dF(y) = s!^{-1}\overline{FGT}(s, \lambda\mu_F)$$

where $\overline{FGT}(s, \lambda\mu_F)$ is the normalised FGT index, defined as

$$\overline{FGT}_F(s, z) = \int_0^z \left(\frac{z-y}{z}\right)^s dF(y). \tag{19}$$

This leads to the following proposition:

Proposition 2 *A necessary and sufficient condition for* $\Delta P_{FG}^R(\lambda) \leq 0$, *given any relative poverty line* $\lambda\mu$ *defined with* $\lambda \in \left[0, \lambda^+\right]$ *and any relative poverty index* $P^R \in \Pi_R^s$, *is:*

$$D_{\widehat{F}}^s(\lambda) - D_{\widehat{G}}^s(\lambda) \geq 0 \quad \forall \lambda \leq \lambda^+. \tag{DSPR}$$

This proposition is similar to the normalised stochastic dominance condition presented in Formby, Smith and Zheng (1999 and 2000) for inequality dominance, except that condition DSPR considers proportions of the mean only up to λ^+. As for welfare comparisons, we can assert that there is no decline in welfare in moving from distribution F to distribution G if:

$$\Delta U_{FG} = \int_0^a u(y)\, dG(y) - \int_0^a u(y)\, dF(y) \geq 0. \tag{20}$$

This leads to the next proposition.

Proposition 3 *A sufficient condition for $\Delta U_{FG} \geq 0$, for all $U \in \Omega^s$, is:*

$$\begin{array}{c} D_F^s(y) - D_G^s(y) \geq 0 \quad \forall y \in [0,a]. \\ \textit{and, if } s \geq 3, \\ D_F^i(a) - D_G^i(a) \geq 0 \quad \forall i \in \{2,...,s-1\}. \end{array} \qquad \textit{(DSU)}$$

Notice that, for $s = 1$ and 2, the dominance conditions for comparisons of absolute poverty and for welfare comparisons are identical if we let $z^+ = a$. If we considered only those functions $u(y)$ for which $u'(y) = 0$ when $y \geq a$, then this similarity would automatically be extended to all orders of stochastic dominance. This would require that the utility "bliss" point be achieved at the maximum income a, and that further increases in incomes passed a would be socially worthless. The results in the last proposition are not new, being already well known in the financial economics literature. Whitmore (1970) introduced the concept of third-order dominance, and Thistle (1993) generalized the concept to all orders of dominance (his results are similar to those of DSU).

For inequality comparisons, we can establish that there is no increase in inequality on passage from distribution F to distribution G if:

$$\Delta I_{FG}^U = I_G^U - I_F^U \leq 0. \tag{21}$$

Using the transformed distributions $\widehat{F}$ and $\widehat{G}$, we can define analogous criteria for stochastic dominance in inequality comparisons.

Proposition 4 *$\Delta I_{FG} \leq 0$ for all $I \in \Upsilon^s$, $s \geq 2$, if:*

$$\begin{array}{c} D_{\widehat{F}}^s(\gamma) - D_{\widehat{G}}^s(\gamma) \geq 0 \quad \forall \gamma \in [0,\alpha], \\ \textit{and, if } s \geq 4, \\ D_{\widehat{F}}^s(\alpha) - D_{\widehat{F}}^s(\alpha) \geq 0 \quad \forall i \in \{3,...,s-1\}, \end{array} \qquad \textit{(DSI)}$$

where $\alpha = \max\left[\frac{a}{\mu_F}, \frac{a}{\mu_G}\right]$.

This is again similar to the normalised stochastic dominance condition of Formby, Smith and Zheng (1999 and 2000). Condition DSI only applies, however, to the limited range for γ of $[0,\alpha]$, and for $s > 3$ it also includes tests at the limit α for the conditions at previous orders[5], $i = 3, ..., s-1$.

[5]The condition in Formby *et. al.* (2000) does not require a test at α for the lower orders of dominance. We will consider such tests in the next section.

Note also that proposition DSI is found in Atkinson (1970) for $s = 2$, which is linked to Lorenz dominance. Shorrocks and Foster (1987) and Davies and Hoy (1994) perform a detailed analysis of third-order dominance in the case of intersecting Lorenz curves, which is equivalent to checking the simpler DSI condition for $s = 3$.

Notice also that we have excluded $s = 1$ in Proposition 4. As alluded to before, this is because first-order (unrestricted) inequality dominance is not possible. To see this, consider Figure 1, where $D^1_{\widehat{F}}(\gamma) = \widehat{F}(\gamma)$. Since $\int_0^\infty \gamma d\widehat{F}(\gamma) = \int_0^\infty \gamma d\widehat{G}(\gamma) = 1$ by definition of the normalised distributions $\widehat{F}(\gamma)$ and $\widehat{G}(\gamma)$, the areas between the dominance curves $D^1_{\widehat{F}}(\gamma)$ and $D^1_{\widehat{G}}(\gamma)$ and the vertical axis must be equal to 1. This is shown on Figure 1, where $A + B = B + C = 1$. It follows from this that the inequality dominance curves $D^1_{\widehat{F}}(\gamma)$ and $D^1_{\widehat{G}}(\gamma)$ must cross at some point γ^+ (shown on Figure 1), and that it therefore cannot be that one of the curves lies always above the other one for all $\gamma \in [0, \alpha]$.

4 Welfare, Inequality, and Poverty Dominance

We now wish to use the dominance criteria DSPA, DSPR, DSU and DSI to establish equivalencies between comparisons of poverty and comparisons of welfare and inequality.

A first difference between poverty dominance criteria and welfare and inequality dominance criteria is that the latter are generally more demanding, since we need to test a series of conditions at a or α at orders less than s. This difference arises from the fact that we implicitly assume the first $s - 1$ derivatives of the poverty indices to vanish at the poverty line, because of the definition of poverty indices (in particular, $p(y, z) = 0$ when $y > z$) and of the $\widehat{C}^s$ continuity assumption. It is natural not to impose an analogous nullity condition for social welfare indices, which explains the presence of conditions at a or α at orders less than s.

Furthermore, it is likely that $z^+ \neq a$, which renders the domain of the tests for poverty dominance narrower or wider than that for social welfare or inequality dominance. However, if we perform both the poverty and the social welfare/inequality dominance tests on the unrestricted domain $[0, \infty)$ rather than on $[0, z^+]$ or $[0, a]$, we can establish certain equivalencies between comparisons of absolute poverty and welfare, on the one hand, and comparisons between relative poverty and inequality, on the other. For this, we draw on the work of Fishburn (1980), who establishes a link between stochastic dominance criteria of various orders and the moments of distribution functions when these functions are such that $a \to \infty$. We generalize these results to all distribution functions belonging to $\Im$ and then adapt them to our analysis. Unlike Fishburn (1980), we also establish a link between s-order dominance conditions and dominance at a or α at lower orders.

To do this, we must first define a binary relationship $>_s^*$ on $\{D_F(x) \mid F \in \Im\}$, where $D_F(x)$ is the sequence $D_F(x) = [D_F^1(x), D_F^2(x), D_F^3(x), ...]$.

Definition 1 *$D_F(x) >_s^* D_G(x)$ if and only if $D_F^i(x) - D_G^i(x) > 0$ for the smallest $i \leq s$ such that $D_F^i(x) \neq D_G^i(x)$.*

Second, we must decompose $D_F^s(x)$ to establish a link between the different orders of dominance conditions.

Lemma 5 *$\forall x \in [a, \infty)$, we have $D_F^s(x) = \sum_{i=1}^{s} \frac{(x-a)^{s-i}}{(s-i)!} D_F^i(a)$.*

Using lemma 5, we can establish the following link between tests of s-order stochastic dominance and lower-order tests:

Theorem 6 *If $F, G \in \Im$, then $D_F(a) >_s^* D_G(a)$ if $D_F^s(x) - D_G^s(x) \geq 0\ \forall x \in \Re_+$.*

This establishes a link between s-order dominance conditions over $\Re_+$ and dominance conditions at the limit a of the distribution at some lower order. Note from Lemma 5, however, that when $D_F(a) >_s^* D_G(a)$, it must be that $\lim_{x\to\infty}[D_F^i(x) - D_G^i(x)] \geq 0$ for all $i \leq s$. We thus have:

Corollary 7 *Given that $F, G \in \Im$, if $F \neq G$ and if $D_F^s(y) - D_G^s(y) \geq 0\ \forall y \in \Re_+$, then $\lim_{x\to\infty}[D_F^i(x) - D_G^i(x)] \geq 0$ for all $i \leq s$.*

Corollary 7 says that if the unrestricted dominance condition at order s is met, then the lower-order conditions at $x \to \infty$ are also met. Hence, extending the domain of the $s-$order test $D_F^s(x) - D_G^s(x) \geq 0$ over the unrestricted positive real domain of x avoids checking conditions at orders lower than s.

We noted at the end of Section 2 that, when $s \to \infty$, only the lowest income mattered for poverty dominance over a range of poverty lines bounded above by a finite maximum poverty line. This result no longer obtains, however, if the range of possible poverty lines extends to infinity. Indeed, the result in corollary 7 indicates that, if we were to obtain unrestricted $s-$order dominance on $\Re_+$, we would implicitly need weak dominance for all orders i lower than s (*i.e.,* $D_F^i(x) - D_G^i(x) \geq 0, i = 1, ..., s-1$) at the point x tending to infinity. All incomes would matter in fulfilling those lower orders of dominance conditions, not only the lowest one.

The result of Corollary 7 leads to further links between absolute poverty and welfare at every order of dominance[6]:

Proposition 8 $\Delta U_{FG} \geq 0$ *for all* $U \in \Omega^s$*, if:*

$$D_F^s(y) - D_G^s(y) \geq 0 \quad \forall y \in [0, \infty). \qquad \text{(DSU')}$$

When testing for social welfare dominance, condition DSU' has the advantage of being a simpler and weaker condition than DSU. By Lemma 5, DSU clearly implies DSU'. DSU' does not, however, necessarily imply DSU. By Theorem 6, all we can say is that if $D_F^s(x) - D_G^s(x) \geq 0, \forall x \in \Re^+$ (and thus if DSU' holds), then $D_F^i(a) - D_G^i(a)$ for the lowest i such that $D_F^i(x) \neq D_G^i(x)$, but not necessarily for the other higher values of i (as required by DSU). Therefore, extending the test range for $s-$order social welfare dominance *increases* the ranking power of the procedure: there are distributions that can be ordered using DSU', but which will fail to be ordered using DSU. A result similar to Proposition 8 obtains for inequality dominance:

Proposition 9 $\Delta I_{FG} \leq 0$ *for all* $I \in \Upsilon^s$*,* $s \geq 2$*, if:*

$$D_{\widehat{F}}^s(\gamma) - D_{\widehat{G}}^s(\gamma) \geq 0 \quad \forall \gamma \in [0, \infty). \qquad \text{(DSI')}$$

Again, condition DSI' is a weaker sufficient condition for inequality dominance than condition DSI in Proposition 4. Using these results it is possible to establish equivalencies between poverty comparisons and welfare and inequality comparisons.

Proposition 10 *Given two distributions* F *and* $G \neq F$*, if* $\Delta P_{FG}^A(z) \leq 0$ *for any index* $P^A \in \Pi_A^s$ *and any threshold* $z \in [0, \infty)$*, then* $\Delta U_{FG} \geq 0$ *for any index* $U \in \Omega^s$*.*

[6]Fishburn's (1980) theorem allows to draw such links only for the first three orders of dominance.

We should note here that Foster and Shorrocks (1988b) obtain similar results for dominance orders $s = 1, 2$ and 3 applied to the FGT class of poverty indices. Using Fishburn's theorem (1980), Foster and Shorrocks' (1988b) results can easily be generalized to poverty indices of the classes Π_A^1, Π_A^2 and Π_A^3. The additional contribution of Theorem 6 and Corollary 7 is to generalize such results to any order of stochastic dominance.

A similar result holds for relative poverty and inequality comparisons.

Proposition 11 *Given two distributions F and $G \neq F$, if $\Delta P_{FG}^R(\lambda) \leq 0$ for any index $P^A \in \Pi_R^s$ and for any relative poverty threshold $\lambda \in [0, \infty)$, then $\Delta I_{FG} \leq 0$ for any index $I \in \Upsilon^s$.*

Again, note that Foster and Shorrocks (1988c) develop a similar result for FGT indices and for $s = 2$ and 3. Theorem 6 and corollary 7 allow us to expand the equivalence between comparisons of relative poverty and comparisons of inequality to all orders of stochastic dominance and to the larger classes Π_R^s of relative poverty indices.

The last two propositions establish a clear link between absolute poverty and welfare comparisons, on the one hand, and comparisons of relative poverty and inequality, on the other. When no maximum poverty line is specified, robust poverty comparisons for the Π_A^s classes of indices are in fact robust welfare comparisons for the Ω^s classes of indices. Moreover, robust relative poverty comparisons for the Π_R^s classes of indices over an unrestricted range of poverty lines are in fact robust comparisons of inequality for the Υ^s classes of indices. In this framework, it becomes possible to treat a ranking of absolute poverty indices as a ranking of welfare indices, and a ranking of relative poverty indices as a ranking of inequality indices when the poverty lines can extend to infinity.

5 Censored Welfare and Inequality Comparisons

Poverty comparisons typically focus, of course, only on a bottom segment of the income distribution. If we specify a finite ceiling on admissible poverty lines, is there still a link between poverty comparisons and welfare and inequality comparisons?

In the manner of Foster and Shorrocks (1988b), let us establish first a link between absolute poverty comparisons and censored welfare comparisons. Assume a certain income ceiling, y^+, at which an analyst wishes to censor the distribution of incomes (the analyst is not interested in the portions of the incomes that exceed this ceiling). Denote by F^* this censored income distribution, such that:

$$\begin{array}{ll} y^* = y, & y \leq y^+, \\ y^* = y^+, & y > y^+. \end{array} \tag{22}$$

G^* is defined similarly. This leads to the following proposition, which links absolute poverty dominance to welfare dominance of censored income distributions.

Proposition 12 *If $\Delta P_{FG}^A(z) \leq 0$ for any index $P^A \in \Pi_A^s$, $s = 1$ or 2, and given any threshold $z \in [0, y^+]$, then $\Delta U_{F^*G^*} \geq 0$ for the censored income distributions F^* and G^* and for any index $U \in \Omega^s$, $s = 1$ or 2.*

Note here that the preceding proposition does not apply to dominance of orders $s \geq 3$. Given that the maximum poverty line is y^+, the sub-domain of the distribution function on which we are testing for dominance is no longer $\Re_+$. In this framework, Theorem 6 and Corollary 7 no longer obtain. Using a framework which differs from ours, Foster and Shorrocks (1988b) proves a similar result, but only for poverty indices of the FGT class. Proposition 12 generalizes this result to all indices in the classes Π_A^1 and Π_A^2. Foster and Shorrocks (1988b) also show an equivalence for $s = 3$. However, for that equivalence they require testing for $\Delta D_{F^*G^*}^2(y^+) \geq 0$. Referring back to Proposition 3, it is clear that adding such lower-order tests at the boundary (y^+) allows extending the equivalencies of Proposition 12 to all orders of dominance. We can thus conclude that, under certain conditions, a measure of absolute poverty may be treated as a measure of welfare for a censored distribution of income.

Let us now introduce a link between comparisons of relative poverty and comparisons of censored inequality. Consider the classes $\Upsilon^{*s} \subset \Upsilon^s$ of inequality indices, $I^U(\gamma^+)$, defined by (13) and (14) with an isoelastic function $u(y) = g(y; \mu)$ such that:

$$g(y; \mu) = \begin{cases} y, & \text{if} \quad y \leq \gamma^+\mu, \\ \gamma^+\mu, & \text{if} \quad y > \gamma^+\mu. \end{cases} \tag{23}$$

For a given mean income, the variable γ^+ represents the proportion of mean income up to which we consider incomes to be relevant in comparing inequality. An index $I(\gamma^+)$ of the class Υ^{*s} is thus not sensitive to occurrences affecting incomes above $\gamma^+\mu$, so long as mean income is unaffected. This effectively corresponds to Pigou-Dalton and generalised transfer principles whose application is limited to the bottom part of the distribution. As with the Pigou-Dalton principle of transfers, this new limited transfer principle stipulates for $s = 2$ that inequality must increase if a mean-preserving transfer occurs from an individual whose income lies below $\gamma^+\mu$ to someone with a higher income. If such a transfer occurs among persons whose incomes are (and remain) greater than $\gamma^+\mu$, the index remains unchanged. Since the new indices do not respect strictly the Pigou-Dalton principle of transfers, we can treat them as indices of censored inequality (in analogy to absolute poverty indices, which as shown above can be thought of as indices of censored social welfare). The class of inequality indices Υ^{*s}, $s = 1, 2, \ldots,$ is thus defined as:

$$\Upsilon^{*s} := \left\{ I^U(\gamma^+) \mid U \in \Omega^s,\, u(y) = v\left[g(y;\mu)\right], \text{ and } v \text{ is isoelastic} \right\}. \tag{24}$$

Note that, unlike for the class Υ^1, there will exist a meaningful possibility of ranking distributions unambiguously over the class Υ^{*1}. We may now state stochastic dominance criteria for these new classes of censored inequality indices, Υ^{*s}.

Proposition 13 $\Delta I_{FG}(\gamma^+) \leq 0$ *for all* $I(\gamma^+) \in \Upsilon^{*s}$*, if:*

$$\begin{array}{c} D^s_{\widehat{F}}(\gamma) - D^s_{\widehat{G}}(\gamma) \geq 0 \quad \forall \gamma \in [0, \gamma^+], \\ \textit{and, if } s \geq 3, \\ D^i_{\widehat{F}}(\gamma^+) - D^i_{\widehat{G}}(\gamma^+) \geq 0 \quad \forall i \in \{2, 3, ..., s-1\}. \end{array} \tag{DSI*}$$

For $s = 1, 2$, this leads straightforwardly to the following link between the measurement of relative poverty and censored inequality:

Proposition 14 *If* $\Delta P^R_{FG}(\lambda) \leq 0$*, for any index* $P^R \in \Pi^s_R$*,* $s = 1$ *or* 2*, and for any threshold* $\lambda \in [0, \gamma^+]$*, then* $\Delta I_{FG}(\gamma^+) \leq 0$ *for any index* $I(\gamma^+) \in \Upsilon^{*s}$*.*

This result implies that we can always treat a ranking of relative poverty at order $s = 1, 2$ as a ranking of censored inequality, that is, inequality judgements that focus only on the lower portions of incomes. For higher orders, Π^s_R regroups indices that effectively also act as indices of censored inequality.

These indices cannot, however, be derived from an additive social welfare formulation for which the individual utility functions are isoelastic (as in (24)) and from the traditional equally-distributed-equivalent income procedure of equation (13). Π_R^s dominance rankings for $s \geq 3$ cannot therefore be linked to censored inequality rankings, as done in Proposition 14 above for $s = 1, 2$. But the indices in Π_R^s are homogeneous of degree 0 in incomes, and they do obey weakly the required $s-$order generalised transfer principles. Since they restrict attention to incomes that belong to $\left[0, \lambda^+\mu\right]$, they can also be thought of as censored inequality indices. Given that relative poverty indices focus only on the bottom part of the distribution, it is easier to obtain a robust relative poverty ranking between different distributions than a robust unrestricted inequality ranking.

6 Relative Poverty and Restricted Inequality Dominance

Censored or restricted inequality comparisons have in fact been frequently performed for a long time in the empirical literature on the distribution of income. The normative foundations of these comparisons do not appear to be well understood, however, and as we now discuss, the comparisons are often done in a manner which is inconsistent with the stochastic dominance criteria derived above.

The first way in which restricted inequality comparisons are effectively found in the literature is through relative poverty comparisons. The commonest example of this is with comparisons of the proportions of individuals found underneath various proportions of the mean as a relative poverty line. This is given by $D^1_{\widehat{F}}(\gamma)$ for distribution F, and is therefore an instance of a test for first-order restricted inequality dominance. For comparisons at higher orders of stochastic dominance, the criteria simply involve the common use of the normalised FGT indices combined with proportions of the mean as relative poverty lines. Note, however, that for relative poverty comparisons to have a meaningful interpretation in terms of restricted inequality comparisons, the relative poverty line must be set as a proportion of the mean, not as a proportion of the median or of another quantile, as is sometimes done.

The second way in which restricted inequality comparisons implicitly arise in the empirical literature is through the use of the dual (or "p-approach") to stochastic dominance (for more on this, see for instance Davidson and Duclos (2000)). For first-order restricted inequality dominance, this involves the use of ratios of quantiles over the mean (normalised quantiles), ratios of quantiles (*e.g.,* the 25^{th}-percentile over the median, or the ratio of the 75^{th}-percentile over the 25^{th}percentile), or differences between normalised quantiles ("normalised interquantile distances").

To see how such comparisons relate to the first-order inequality dominance criteria derived above, consider Figure 2. On the horizontal axis, we find the proportion γ of the mean used as a relative poverty line to compare relative poverty in distributions F and G. On the vertical axis, we find the proportion of individuals below those relative poverty lines in F and in G. Let $Q_F(p)$ be the inverse of the distribution function F, such that $Q_F(p) = F^{-1}(p)$. $Q_F(p)$ is thus the $p-$quantile of distribution F. Let $Q_G(p)$ be defined analogously, and denote by $\Gamma(p) = Q(p)/\mu$ the normalised quantiles. By definition, we thus have that $F(\Gamma_F(p)\mu_F) = F(Q_F(p)) = p$. For first-order relative poverty dominance of G over F, we thus require that $F(\gamma\mu_F) \geq G(\gamma\mu_G)$ for a range of γ, which in Figure 2 is obtained for all positive γ until γ^+. Equivalently, this can be tested by comparing normalised quantiles and by checking that $\Gamma_F(p) \leq \Gamma_G(p)$ for a range of p, which in Figure 2 is obtained for all positive p until p^+. Clearly, $F(\gamma^+\mu_F) = G(\gamma^+\mu_G) = p^+$. Hence, we have restricted first-order inequality dominance of G over F up to γ^+ and p^+.

The comparison of normalised quantiles in the empirical income distribution literature is thus consistent with the first-order restricted inequality dominance criteria developed above. To lead to a robust ranking of censored inequality, such comparisons must, however, be done over a range of p values, starting from zero.

Ratios of quantiles and differences between normalised quantiles, are not, however, generally consistent with inequality dominance rankings. To see this, note in Figure 2 that although distribution F has robustly more censored inequality than G, the ratio of the median over the 25^{th}-percentile is *less* in F than in G: $\Gamma_F(0.25)/\Gamma_F(0.5) > \Gamma_G(0.25)/\Gamma_G(0.5)$. This wrongly suggests that there is *less* inequality in F than in G. Similarly, the normalised interquantile distance $\Gamma(0.5) - \Gamma(0.25)$ is also lower in F than in G, suggesting again wrongly that inequality is lower in F. Moreover, since first-order restricted inequality dominance (up to γ^+) implies higher-order restricted inequality dominance (at least up to γ^+), quantile ratios and normalised interquantile distances are inconsistent not only with first-order restricted inequality dominance, but also with any higher–order concept of restricted or unrestricted inequality dominance (and hence with Lorenz dominance). These measures should therefore be avoided.

7 Relative Poverty and Restricted Inequality Dominance: A Geometric Analysis

This section further develops the link between poverty and inequality by illustrating graphically how a focus on relative poverty and censored inequality comparisons gives us a greater ranking power than a consideration of unrestricted inequality comparisons. To do this, assume a society comprising three individuals and who share a total income of 3μ. In what follows, when a distribution x is more unequal than a distribution y, for all $I \in \Upsilon^s$, we write $xI^s y$.

Refer now to Figure 3 and assume that the initial income distribution is given by point a in the interior of the simplex. Let us now determine those distributions for which there exists a robust ranking of inequality indices of class Υ^2, *i.e.* distributions for which the Lorenz curves do not intersect those of distribution a. We know that permutations of the three individuals' incomes yield distributions for which inequality is exactly the same as for distribution a. If we connect these points with line segments we obtain an irregular hexagon. All points on these line segments represent income distributions with less inequality than distribution a, except those points representing permutations of the incomes in a. The reason for this is relatively straightforward: at any point on a given line segment, a progressive transfer of income is occurring between two individuals while that of the third individual is constant. The set of all points in the interior of the hexagon represents income distributions with less inequality than a. Indeed, moving into the interior of the hexagon, one approaches the point (μ, μ, μ), which portrays a perfectly egalitarian distribution of incomes.

We have just provided a description of the set of distributions whose Lorenz curves dominate the Lorenz curve of distribution a (aI^2y). Let us now find distributions whose Lorenz curves are dominated by that of distribution a (yI^2a). These distributions are represented by the regions in the arrowheads situated at the extremities of the simplex. In fact, distribution a may be obtained from any one of these distributions through the simple implementation of progressive income transfers. Finally, there are distributions whose Lorenz curves intersect that of distribution a (?) . We know that these distributions are dominated by distributions situated on the line segments forming the hexagon. However, since these distributions on the segments dominate distribution a, we cannot say anything concerning comparisons of inequality between distributions situated in the regions designated (?) and distribution a.

There are two ways in which we can increase the size of the set of distributions that can be ranked with a. The first is by following Foster and Shorrocks (1988b and c) and Davies and Hoy (1994) and by restricting the analysis to indices of class Υ^3. Inequality indices of class Υ^3 are those which are sensitive to composite transfers. A beneficial composite transfer is one between four individuals, i, j, k, and l, having incomes $y_i < y_j \leq y_k < y_l$. The transfer is from j to i and from k to l, such that the variance of the distribution does not change. Note that the first transfer is a progressive transfer, while the second is not. This approach increases the number of distributions that can be subject to a robust ranking with respect to a through the addition of the regions denoted $(+)$ and $(-)$ in Figure 4.

To understand this diagram, refer to the definition of the variance for a distribution of income between three persons:

$$3 \cdot \sigma^2 = (y_1 - \mu)^2 + (y_2 - \mu)^2 + (y_3 - \mu)^2 . \tag{25}$$

The set of points whose variance is equal to the variance of a is thus a sphere centred on (μ, μ, μ) and passing through a. The intersection of this sphere with the simplex gives us the circle illustrated in Figure 4. The regions denoted $(+)$ depict distributions for which the Lorenz curves intersect the Lorenz curve of distribution a and are such that aI^3y. The regions represented by $(-)$ depict distributions for which the Lorenz curves intersect the Lorenz curve of distribution a and are such that yI^3a. To understand this, notice that all distributions on the circle between points a and b, between points c and d, and between points e and f, are such that aI^3y. These loci are, in fact, obtained by redistributing some of the income of the person with a median income towards the richest and the poorest person without increasing the variance of the distribution. Conversely, the distributions located between points a and f, between points b and c, and between points d and e are such that yI^3a. These distributions arise from shifting some income from the wealthiest person, and some from the poorest person, toward the person in the middle without increasing the variance. We know that distribution a can be obtained from these distributions through implementation of a composite transfer. Inequality is thus greater in these distributions for indices of class Υ^3.

We now wish to illustrate how restricting the analysis to measures of the class Υ^{*2} increases the number of distributions that can be robustly compared with a. Though convex, social indifference curves corresponding to indices of the class Υ^{*2} are not strictly convex, unlike social indifference curves corresponding to indices sensitive to the entire domain of income distributions. Let us examine the case of a total income 2μ being split between two persons. Figure 5 illustrates a social indifference curve corresponding to an inequality index sensitive to the entire distribution. Figure 6 illustrates the analogous curve for an index belonging to Υ^{*2}. Notice that, in this latter case, the set of income distributions for which each individual has an income greater than $\gamma^+\mu$ is equivalent, from a social perspective, to the egalitarian distribution in which each has income μ.

Returning to the case of three individuals, the flat segment of the social indifference curve now corresponds to the regions situated in the interior of the dotted lines in Figure 7. These distributions are preferred to a because they are equivalent, from a social perspective, to the egalitarian distribution under social preferences corresponding to inequality indices of class Υ^{*2}. As this triangle includes points (regions denoted $(+)$) corresponding to income distributions for which the Lorenz curves intersect the Lorenz curve of distribution a, the restriction to indices of the class Υ^{*2} has enhanced our ability to generate robust rankings of these distributions. Given that these indices are sensitive only to the lower income portions in the income distribution, the optimal point, from a social perspective, is not unique.

The set of points located within the dotted triangle of Figure 7 is thus equivalent to the point (μ, μ, μ) from society's point of view. This triangle corresponds to the flat section on the indifference curve in Figure 6. Moreover, regions denoted $(-)$ correspond to income distributions such that $yI^{*2}a$, for which the Lorenz curves intersect that of distribution a. Given that inequality indices belonging to Υ^{*2} are only sensitive to changes at the bottom of the distribution, we obtain a robust ranking even if the Lorenz curves intersect. To see this, bear in mind that all points situated on the line segment passing through a and f, and located between the dotted lines, are equivalent to point a from a social perspective. Again, this is attributable to the fact that the social indifference surfaces are not strictly convex for the class Υ^{*2}. These points are located on a flat segment of this surface. Conversely, points located on any line segment parallel to that passing through a and f, situated to the north-west of it, and bracketed by the two dotted lines, constitute loci of distributions which are equivalent but entail more inequality than a, since they belong to an indifference surface below that of a.

Note that for every distribution a there exists a range of sufficiently small γ^+ such that the power of ranking a over the class Υ^{*2} is greater than the power of ranking over the class Υ^2. This requires that γ^+ be small enough for the distribution a to be restricted to a flat segment of the social indifference surface. The maximal value of γ^+ for which this ranking power is greater for Υ^{*2} than for Υ^2 is obtained when the dotted lines that form a triangle in Figure 7 pass through a. This maximal value of γ^+ is illustrated in Figure 8. For greater values of γ^+ than the one shown in Figure 8, the ranking power of Υ^{*2} for a is identical to that of Υ^2.

Lastly, recall that a robust relative poverty ranking for class Π_R^1 may also be interpreted as a robust inequality ranking for the class Υ^{*1}. Recall that it is not possible to rank distributions robustly across all inequality indices that are members of Υ^1. Intuitively, this is because it is impossible for every normalised quantile to be greater in one distribution than in another: it cannot be that everyone's share of total income be greater in one distribution than in another. Such a ranking may, however, be possible for class Υ^{*1}. To see this graphically, refer to Figure 9 and consider the income distribution represented by point a. No other distribution in the simplex can dominate a or be dominated by a for a Υ^1 inequality dominance ranking. However, what of indices belonging to class Υ^{*1}? First, we know that distributions within the dotted triangle dominate distribution a, because under these distributions no one has an income below $\gamma^+\mu$. We also know that distributions represented by points on the line segments bc, ef, and hi are equivalent to distribution a for these indices. Indeed, points on the segment bc are generated from distribution a by redistributing income between individuals whose income is greater than $\gamma^+\mu$ before and after the redistribution. Since indices of the class Υ^{*1} are not sensitive to these transfers, these distributions are equivalent to a.

Points on the segments ef and hi simply represent distributions symmetric to those on segment bc. Points located between the segment bc and the dotted triangle (+) dominate distribution a, since they are obtained by increasing the income of a person situated below $\gamma^+\mu$ while maintaining the remaining incomes above that line. Points between the segment bc and the north-east border of the simplex (−) are dominated by distribution a, since they are obtained by redistributing some share of income from the person below the $\gamma^{+}\mu$ line to those whose income is already above $\gamma^{+}\mu$. Points on the segment cd (excluding points c and d) cannot be ranked relative to a in a robust manner, since they imply redistribution between two individuals whose income is below $\gamma^{+}\mu$. Points situated between the segment ce and the dotted triangle (?) dominate the distributions on the segment ce. However, they cannot be ranked relative to a, since points on the segment ce cannot be given a robust ranking relative to a. Points between ce and the edge of the triangle cde are dominated by distributions on the segment ce. However, they cannot be ranked with respect to a, since points on the segment ce cannot be subject to a robust ranking with respect to a. Finally, points on the segment cd, except the point c, are dominated by a (and c), since they imply a transfer from a person whose income is below $\gamma^{+}\mu$ to a person whose income is above $\gamma^{+}\mu$, while maintaining the third person's income constant. The set of points between the triangle cde and the border of the simplex can be reached by transferring from a person whose income is below $\gamma^{+}\mu$ toward another whose income is above $\gamma^{+}\mu$. Since each of these points is dominated by at least one point on the segment cd (or de), they are dominated by a.

Thus we note graphically that by limiting ourselves to censored indices of inequality, we can increase significantly the ranking power of ordinal inequality comparisons. This is most striking for first-order restricted inequality dominance, by which the set of distributions located in sections represented by the regions denoted (+) or (−) in Figure 9 are robustly ranked with respect to distribution a, but for which a Υ^{1} ranking would not have been possible for any of the distributions in the simplex.

8 Conclusion

This paper extends some of the theoretical links between poverty comparisons and comparisons of welfare and inequality. Assumptions of various degrees of continuity of poverty indices at the poverty line make poverty rankings generally easier to obtain than welfare or inequality rankings since this avoids the need to check dominance criteria at lower orders of dominance. Robust rankings are also naturally easier to obtain for poverty than welfare when the maximum poverty line is set below the maximal income value. Extending from the top income to infinity the upper income bound for dominance testing increases the ranking power of social welfare and inequality dominance criteria. This procedure also has the virtue of removing the arbitrariness of choosing a maximum value as the upper bound of welfare and inequality dominance test ranges.

When the maximum poverty line is set equal to the upper income bound, absolute poverty and welfare dominance criteria become equivalent for first- and second-order dominance. When the maximum relative poverty line extends to the upper income bound, relative poverty and inequality dominance criteria become equivalent for the first three orders of dominance. If we do not restrict the maximum poverty line, and let it tend toward infinity, the above criteria become equivalent for all orders of stochastic dominance. For some lower orders of dominance, these links between poverty, welfare and inequality are also valid when the maximum poverty line falls below the maximum income bound, so long as social welfare and inequality judgements are "censored", that is, so long as they do not take into account the portions of income that exceed a particular threshold.

The paper particularly emphasises the links between relative poverty and inequality at arbitrary orders of stochastic dominance. In particular, first-order relative poverty orderings can be usefully interpreted as first-order restricted inequality orderings. Such first-order inequality orderings are in fact frequently found in the empirical literature on the distribution of income. The tools used to perform such orderings include, *inter alia,* proportions of individuals underneath various proportions of the mean or the median, quantiles normalised by the mean, ratios of quantiles, and mean-normalised interquantile distances. We find that comparisons of normalised quantiles and of proportions of individuals underneath proportions of the mean are consistent and provide appropriate first-order inequality dominance criteria, but that the other tools do not, and that their use is therefore inconsistent with the search for inequality dominance of *any* order. A geometric analysis finally illustrates how censoring inequality indices can increase significantly the ranking power of inequality comparisons, and how this compares with other ways of extending the ranking power of stochastic dominance criteria.

Appendix

Proof of proposition 1. In order to prove proposition 1, we first need to integrate equation (1) by parts:

$$\int_0^a p_A(y,z)\,dF(y) = p_A(y,z)\,D_F^1(y)\Big|_0^a - \int_0^a \frac{\partial p_A(y,z)}{\partial y} D_F^1(y)\,dy. \tag{26}$$

We know that, by definition of the domain, $D_F^1(0) = 0$. Furthermore, assumption A1 postulates that $p_A(a,z) = 0$ since $z \leq a$. The first term on the r.h.s. is thus nil. Consequently, we have:

$$\int_0^a p_A(y,z)\,dF(y) = -\int_0^a \frac{\partial p_A(y,z)}{\partial y} D_F^1(y)\,dy. \tag{27}$$

Now, assume that, for some $i \in \{2,...,s-1\}$, we have:

$$\int_0^a p_A(y,z)\,dF(y) = (-1)^{i-1} \int_0^a \frac{\partial^{i-1} p_A(y,z)}{\partial y^{i-1}} D_F^{i-1}(y)\,dy. \tag{28}$$

Integrating by parts, we obtain:

$$\begin{aligned}\int_0^a p_A(y,z)\,dF(y) &= (-1)^{i-1}\left\{ \frac{\partial^{i-1} p_A(y,z)}{\partial y^{i-1}} D_F^i(y)\Bigg|_0^a \right. \\ &\quad \left. - \int_0^a \frac{\partial^i p_A(y,z)}{\partial y^i} D_F^i(y)\,dy \right\}.\end{aligned} \tag{29}$$

We know that, by definition of the domain, $D_F^i(0) = 0$. Furthermore, the assumption in equation (2) postulates that $p_A(a,y) = 0$ for all $y \geq z$. Consequently, we know that $\frac{\partial^{i-1} p_A(a,z)}{\partial y^{i-1}} = 0$. Finally, since $D_F^i(a) \neq \infty$, the first term in the braces vanishes. Thus, we have:

$$\int_0^a p_A(y,z)\,dF(y) = (-1)^i \int_0^a \frac{\partial^i p_A(y,z)}{\partial y^i} D_F^i(y)\,dy. \tag{30}$$

Equation (14) respects the relationship in (3). Since we have shown that equation (3) implies equation (16), we know that this latter equation is true for all $i \in \{1,2,...,s-1\}$. By assumption, the function $\frac{\partial^{s-1} p_A(y,z)}{\partial y^{s-1}}$ is continuous and either differentiable or piecewise differentiable. If the function

$\frac{\partial^{s-1}p_A(y,z)}{\partial y^{s-1}}$ has n points at which it is not differentiable, we denote these points y_k, $k=1,2,\ldots n$. Integrating equation (16) by parts, for $i=s-1$, yields:

$$\begin{aligned}\int_0^a p_A(y,z)\,dF(y) &= (-1)^{s-1}\frac{\partial^{s-1}p_A(y,z)}{\partial y^{s-1}}D_F^s(y)\Big|_0^a \\ &\quad -\sum_{k=0}^{n}\int_{y_k}^{y_{k+1}}\frac{\partial^s p_A(y,z)}{\partial y^s}D_F^s(y)\,dy.\end{aligned} \tag{31}$$

We know that, by definition of the domain, $D_F^s(0)=0$. Furthermore, assumption A1 postulates that $p_A(a,y)=0$ for all $y\geq z$. Consequently, we know that $\frac{\partial^{s-1}p_A(a,z)}{\partial y^{s-1}}=0$. The first term in the braces is thus nil. We have:

$$\int_0^a p_A(y,z)\,dF(y)=(-1)^s\sum_{k=0}^{n}\int_{y_k}^{y_{k+1}}\frac{\partial^s p_A(y,z)}{\partial y^s}D_F^s(y)\,dy. \tag{32}$$

Substituting equation (32) into equation (13) yields:

$$\Delta P_{FG}^A=(-1)^s\sum_{k=0}^{n}\left\{\int_{y_k}^{y_{k+1}}\frac{\partial^s p_A(y,z)}{\partial y^s}D_G^s(y)\,dy\right. \tag{33}$$

$$\Delta P_{FG}^A=(-1)^s\sum_{i=0}^{n}\int_{y_k}^{y_{k+1}}\frac{\partial^s p_A(y,z)}{\partial y^s}\left[D_G^s(y)-D_F^s(y)\right]dy. \tag{34}$$

To prove sufficiency we only need to use equation (34) and the definition of the set Π_A^s. We can then conclude that, if we have $D_F^s(y)-D_G^s(y)\geq 0,\ \forall y\leq z^+$, we must have $\Delta P_{FG}^A(z)\leq 0$. In order to establish necessity, we use a function $p_A(y,z)$, the $(s-1)$-st derivative of which is:

$$\frac{\partial^{s-1}p_A(y,z)}{\partial y^{s-1}}=\begin{cases}(-1)^{s-1}\epsilon, & \text{if } y\leq\overline{y},\\ (-1)^{s-1}(\overline{y}+\epsilon-y), & \text{if } \overline{y}<y\leq\overline{y}+\epsilon,\\ 0, & \text{if } y\geq\overline{y}+\epsilon,\end{cases}, \tag{35}$$

where $\overline{y}<z$. Poverty indices whose functions $p_A(y,z)$ have the above-given form for $\frac{\partial^{s-1}p_A(y,z)}{\partial y^{s-1}}$ belong to the class Π^s. This yields:

$$\frac{\partial^s p_A(y,z)}{\partial y^s}=\begin{cases}0, & \text{if } y\leq\overline{y},\\ (-1)^s, & \text{if } \overline{y}\leq y\leq\overline{y}+\epsilon,\\ 0, & \text{if } y\geq\overline{y}+\epsilon.\end{cases} \tag{36}$$

Imagine now that $D_F^s(y) - D_G^s(y) < 0$ on an interval $[\overline{y}, \overline{y} + \epsilon]$, with $\overline{y} < z$ and ϵ arbitrarily close to 0. For $p_A(y, z)$ defined as in (35), $\Delta P_{FG}^A(z)$ is thus positive and poverty increases with a movement from distribution F to distribution G. Hence it cannot be that $D_F^s(y) - D_G^s(y) < 0$ for some $y \in [\overline{y}, \overline{y} + \epsilon]$ when $\overline{y} < z^+$. This proves the necessity of the condition. ■

Proof of proposition 2. In order to prove the sufficiency of the DSPR condition we use assumption A4. Since the function $p_R(y, \lambda\mu_F)$ is homogeneous of degree zero in y and μ_F, we know that $p_R(y, \lambda\mu_F) = p_R(\widehat{y}_F, \lambda)$ and, by extension, $P_F^R = P_{\widehat{F}}^R$. Similarly, we conclude that $P_G^R = P_{\widehat{G}}^R$. We know that $\mu_{\widehat{F}} = \mu_{\widehat{G}} = 1$ and that $p_R(\widehat{y}, \lambda)$ belongs to the class Π^s when we fix $\lambda = z$. Thus, we only need to apply the DSPA criteria to conclude that testing:

$$D_{\widehat{F}}^s(\lambda) - D_{\widehat{G}}^s(\lambda) \geq 0, \quad \forall \lambda \leq \lambda^+, \tag{37}$$

allows us to confirm that $\Delta P_{\widehat{F}\widehat{G}}^A(\lambda) \leq 0$. And, since $p_R(y, \lambda\mu_F) = p_R(\widehat{y}_F, \lambda)$ and $p_R(y, \lambda\mu_G) = p_R(\widehat{y}_G, \lambda)$, we can further conclude that $\Delta P_{FG}^R(\lambda) \leq 0$. To prove the necessity of the condition, we only need consider indices whose $(s-1)$-th derivative is given by:

$$\frac{\partial^{s-1} p_R(y, \lambda\mu)}{\partial y^{s-1}} = \begin{cases} (-1)^{s-1} \frac{\epsilon}{(\phi\mu)^s}, & \text{if } \; y \leq \phi\mu, \\ (-1)^{s-1} \frac{(\phi\mu + \epsilon - y)}{(\phi\mu)^s}, & \text{if } \; \phi\mu < y \leq \phi\mu + \epsilon, \\ 0, & \text{if } \; y \geq \phi\mu + \epsilon, \end{cases} \tag{38}$$

where $\phi < \lambda$, and then proceed as in proposition 1. ■

Proof of proposition 3. In order to prove proposition 3, we first need to integrate equation (11) by parts:

$$\int_0^a u(y)\, dF(y) = u(y)\, D_F^1(y)\Big|_0^a - \int_0^a \frac{du(y)}{dy} D_F^1(y)\, dy. \tag{39}$$

Now, assume that, for some $i \in \{2, ..., s-1\}$, we have:

$$\begin{aligned} \int_0^a u(y)\, dF(y) &= \sum_{j=1}^{i-1} \left[(-1)^{j-1} u^{(j-1)}(y)\, D_F^j(y)\Big|_0^a \right] \\ &\quad + (-1)^{i-1} \int_0^a u^{(i-1)}(y)\, D_F^{i-1}(y)\, dy, \end{aligned} \tag{40}$$

where $u^{(0)}(y) = u(y)$ and $u^{(j)}(y) = d^j u(y)/dy^j$. Integrating by parts, we obtain:

$$\int_0^a u^{(i-1)}(y)\, D_F^{i-1}(y)\, dy = u^{(i-1)}(y)\, D_F^i(y)\Big|_0^a - \int_0^a u^{(i)}(y)\, D_F^i(y)\, dy. \quad (41)$$

Substituting equation (41) into equation (40) yields:

$$\begin{aligned}\int_0^a u(y)\, dF(y) &= \sum_{j=1}^{i}\left[(-1)^{j-1}\, u^{(j-1)}(y)\, D_F^j(y)\Big|_0^a\right] \quad (42)\\ &\quad + (-1)^i \int_0^a u^{(i)}(y)\, D_F^i(y)\, dy.\end{aligned}$$

Equation (39) respects the relashionship in (40). Since we have shown that equation (40) implies equation (42), we know that this latter is true for all $i \in \{2, ..., s-1\}$. By assumption, the function $u^{(s-1)}$ is continuous and either differentiable or piecewise differentiable. If the function $u^{(s-1)}$ has n points at which it is not differentiable, we denote these points y_k, $k = 1, 2, ..., n$. Integrating equation (42) by parts, for $i = s-1$, yields:

$$\begin{aligned}\int_0^a u(y)\, dF(y) &= \sum_{j=1}^{s}\left[(-1)^{j-1}\, u^{(j-1)}(y)\, D_F^j(y)\Big|_0^a\right] \quad (43)\\ &\quad + (-1)^s \sum_{k=0}^{n}\int_{y_k}^{y_{k+1}} u^{(s)}(y)\, D_F^s(y)\, dy.\end{aligned}$$

We know that $D_F^j(0) = D_G^j(0) = 0$ for all j and that , puisque $D_F^1(a) = D_G^1(a) = 1$. We thus have:

$$\begin{aligned}\Delta U_{FG} &= \sum_{i=2}^{s}(-1)^{i-1} u^{(i-1)}(a)\left[D_G^i(a) - D_F^i(a)\right] \quad (44)\\ &\quad + (-1)^s \sum_{k=0}^{n}\int_{y_k}^{y_{k+1}} u^{(s)}(y)\left[D_G^s(y) - D_F^s(y)\right] dy.\end{aligned}$$

For $s = 2$, $\Delta U_{FG} \geq 0$ if $D_F^s(y) - D_G^s(y) \geq 0$, $\forall y \in [0, a]$. For $s \geq 3$, $\Delta U_{FG} \geq 0$ if $D_F^i(a) - D_G^i(a) \geq 0$, $i = 2$ to $s-1$ and $D_F^s(y) - D_G^s(y) \geq 0$, $\forall y \in [0, a]$. This proves the sufficiency of condition DSU. ■

Proof of proposition 4. In order to prove the DSI condition we use an approach similar to that in Atkinson (1970). To do this we use the isoelasticity of the function $u(y)$. Given that $u(y)$ is isoelastic, we can write

$u\left(\frac{y}{\mu_F}\right) = \left(\frac{1}{\mu_F}\right)^r u(y)$, where r represents the elasticity of the function $u(y)$. We then have:

$$\int_0^\alpha u(x)\, d\widehat{F}(x) = \left(\frac{1}{\mu_F}\right)^r \int_0^a u(y)\, dF(y), \tag{45}$$

and thus:

$$u\left(y_{e,\widehat{F}}\right) = \left(\frac{1}{\mu_F}\right)^r u(y_{e,F}). \tag{46}$$

Again, using the isoelasticity of the function u, we conclude that:

$$y_{e,\widehat{F}} = \frac{y_{e,F}}{\mu_F}. \tag{47}$$

Now, we know that $\mu_{\widehat{F}} = 1$ by definition. We thus see that $I_{\widehat{F}} = I_F$, and, similarly, $I_{\widehat{G}} = I_G$. However, since by definition, $\mu_{\widehat{F}} = \mu_{\widehat{G}} = 1$, it is sufficient to test whether $y_{e,\widehat{G}} \geq y_{e,\widehat{F}}$ in order to establish whether $\Delta I_{FG} = I_G - I_F \leq 0$. Given that $u(y)$ is an increasing function, this test is equivalent to verifying that $\Delta U_{\widehat{F}\widehat{G}} \geq 0$, which is, in fact, a welfare comparison for the transformed functions $\widehat{F}$ and $\widehat{G}$. We now can use the DSU condition to recuperate the DSI condition. In addition, we notice that, for $s = 3$, it is no longer necessary to test for $D^2_{\widehat{F}}(\alpha) - D^2_{\widehat{G}}(\alpha) \geq 0$, since, from the definitions of $\widehat{F}$ and $\widehat{G}$, we have $D^2_{\widehat{F}}(\alpha) - D^2_{\widehat{G}}(\alpha) = 0$. Integration by parts of $D^2_{\widehat{F}}(\alpha)$ yields:

$$D^2_{\widehat{F}}(\alpha) = \alpha D^1_{\widehat{F}}(\alpha) - \mu_{\widehat{F}}. \tag{48}$$

And so:

$$D^2_{\widehat{F}}(\alpha) - D^2_{\widehat{G}}(\alpha) = \alpha D^1_{\widehat{F}}(\alpha) - \mu_{\widehat{F}} - \alpha D^1_{\widehat{G}}(\alpha) - \mu_{\widehat{G}}. \tag{49}$$

However, by definition, we have $D^1_{\widehat{F}}(\alpha) = D^1_{\widehat{G}}(\alpha)$ and $\mu_{\widehat{F}} = \mu_{\widehat{G}}$. We may thus conclude that $D^2_{\widehat{F}}(\alpha) - D^2_{\widehat{G}}(\alpha) = 0$. ■

Proof of lemma 5. If $x \geq a$, $D^1_F(x) = D^1_F(a) = 1$. Thus, $D^2_F(x)$ may be rewritten as:

$$D^2_F(x) = D^2_F(a) + \int_a^x D^1_F(y)\, dy. \tag{50}$$

$D^1_F(y)$ being constant for $y \geq a$, the above expression may be rewritten as:

$$D^2_F(x) = D^2_F(a) + (x - a)\, D^1_F(a). \tag{51}$$

Analogously, $D_F^3(x)$ may be written as:

$$D_F^3(x) = D_F^3(a) + \int_a^x D_F^2(y)\,dy. \tag{52}$$

Using the result in equation (51), the above equation becomes:

$$D_F^3(x) = D_F^3(a) + \int_a^x D_F^2(a)\,dy + \int_a^x (y-a)\,D_F^1(a)\,dy, \tag{53}$$

$$D_F^3(x) = D_F^3(a) + D_F^2(a)\int_a^x dy + D_F^1(a)\int_a^x (y-a)\,dy, \tag{54}$$

$$D_F^3(x) = D_F^3(a) + (x-a)\,D_F^2(a) + \frac{(x-a)^2}{2}D_F^1(a). \tag{55}$$

Now, assume that we have:

$$D_F^{s-1}(x) = \sum_{i=1}^{s-1}\frac{(x-a)^{s-1-i}}{(s-1-i)!}D_F^i(a). \tag{56}$$

Consequently, we thus have:

$$D_F^s(x) = D_F^s(a) + \int_a^x D_F^{s-1}(y)\,dy, \tag{57}$$

Using equation (56), the above expression may be rewritten as:

$$D_F^s(x) = D_F^s(a) + \int_a^x \left\{\sum_{i=1}^{s-1}\frac{(y-a)^{s-1-i}}{(s-1-i)!}D_F^i(a)\right\}dy, \tag{58}$$

$$D_F^s(x) = D_F^s(a) + \sum_{i=1}^{s-1}\left\{D_F^i(a)\int_a^x \frac{(y-a)^{s-1-i}}{(s-1-i)!}dy\right\}, \tag{59}$$

$$D_F^s(x) = D_F^s(a) + \sum_{i=1}^{s-1}\frac{(x-a)^{s-i}}{(s-i)!}D_F^i(a), \tag{60}$$

$$D_F^s(x) = \sum_{i=1}^{s}\frac{(x-a)^{s-i}}{(s-i)!}D_F^i(a). \tag{61}$$

Equations (51) and (55) respect the relationship in (56). Since we have shown that equation (56) implies equation (61), we know that this latter equation is true for all $s \geq 2$. ■

Proof of theorem 6. We begin by using the result from lemma 5 to rewrite $D_F^s(x) - D_G^s(x)$, for $x > a$, as:

$$\Delta D_{FG}^s(x) = \sum_{i=2}^{s} \frac{(x-a)^{s-i}}{(s-i)!} \Delta D_{FG}^i(a), \tag{62}$$

where $\Delta D_{FG}^s(x) = D_F^s(x) - D_G^s(x)$ for all s. In the preceding equation, the term $\frac{(x-a)^{s-1}}{(s-1)!} \Delta D_{FG}^1(a)$ disappears, because $\Delta D_{FG}^1(a) = 0$ by definition. Notice now that $\frac{(x-a)^{s-i}}{(s-i)!}$ is of order $O(x^{s-i})$. Consequently, we know that if $\lim_{x\to\infty} \Delta D_{FG}^2(x) \neq 0$ then:

$$sign\left[\lim_{x\to\infty} \Delta D_{FG}^s(x)\right] = sign\left[\Delta D_{FG}^2(a)\right]. \tag{63}$$

If, however, $\Delta D_{FG}^2(a) = 0$ and $\Delta D_{FG}^3(a) \neq 0$, then:

$$sign\left[\lim_{x\to\infty} \Delta D_{FG}^s(x)\right] = sign\left[\Delta D_{FG}^3(a)\right]. \tag{64}$$

Thus, in general, if $\Delta D_{FG}^j(a) = 0$ for all $j < i$ and $\Delta D_{FG}^i(a) \neq 0$, then:

$$sign\left[\lim_{x\to\infty} \Delta D_{FG}^s(x)\right] = sign\left[\Delta D_{FG}^i(a)\right] \text{ for all } s \geq i. \tag{65}$$

Theorem 6 follows directly form this result. ■

Proof of proposition 8. This is obtained by applying corrolory 7 to the DSU condition and testing up to ∞ rather than up to a. ■

Proof of proposition 9. This is obtained by applying corrolory 7 to the DSI condition and testing up to ∞ rather than up to a. ■

Proof of proposition 10. This proposition derives directly from the equivalence between the DSPA condition and the DSU' condition when the maximum poverty line $z^+ \to \infty$. ■

Proof of proposition 11. This proposition follows directly from the equivalence of the DSPR condition and the DSI' condition when the maximum relative poverty line $\lambda^+ \to \infty$. ■

Proof of proposition 12. This proposition derives directly from the definition of censoring in equation (22). Fixing $z^+ = y^+$, we easily see that the DSPA condition is equivalent to the DSU condition. ■

Proof of proposition 13. In order to prove the DSI* condition we use the isoelasticity of the function $v(y)$. Given that $v(y)$ is isoelastic, given that $g(y)$ is homogenous of degree 1 for $y \leq \gamma^+ \mu_F$ and given that μ_F is also homogenous of degree 1 in y, we can write $u\left(\frac{y}{\mu_F}\right) = \left(\frac{1}{\mu_F}\right)^r u(y)$ where r represents the elasticity of the function $v(y)$. We then have:

$$\int_0^a u(x)\, d\widehat{F}(x) = \left(\frac{1}{\mu_F}\right)^r \int_0^a u(y)\, dF(y), \tag{66}$$

and thus:

$$u\left(y_{e,\widehat{F}}\right) = \left(\frac{1}{\mu_F}\right)^r u(y_{e,F}). \tag{67}$$

Again, using the isoelasticity of the function v, we conclude that:

$$y_{e,\widehat{F}} = \frac{y_{e,F}}{\mu_F}. \tag{68}$$

Now, we know that $\mu_{\widehat{F}} = 1$ by definition. We thus see that $I_{\widehat{F}} = I_F$, and, similarly, $I_{\widehat{G}} = I_G$. However, since by definition, $\mu_{\widehat{F}} = \mu_{\widehat{G}} = 1$, it is sufficient to test whether $y_{e,\widehat{G}} \geq y_{e,\widehat{F}}$ in order to establish whether $\Delta I_{FG} = I_G - I_F \leq 0$. Given that $u(y)$ is an increasing function, this test is equivalent to verifying that:

$$\int_0^a u(x)\, d\widehat{G}(x) - \int_0^a u(x)\, d\widehat{F}(x) \geq 0, \tag{69}$$

which is, in fact, a welfare comparison of the transformed functions $\widehat{F}$ and $\widehat{G}$. We can now rewrite the above expression as:

$$\begin{aligned}\int_0^a u(x)\, d\widehat{G}(x) - \int_0^a u(x)\, d\widehat{F}(x) &= \int_0^{\gamma^+} v(x)\, d\widehat{G}(x) \\ &\quad - \int_0^{\gamma^+} v(x)\, d\widehat{F}(x) \\ &\quad + v\left(\gamma^+\right)\left[D^1_{\widehat{F}}\left(\gamma^+\right) - D^1_{\widehat{G}}\left(\gamma^+\right)\right]\end{aligned} \tag{70}$$

Integrating $\int_0^{\gamma^+} v(x)\, d\widehat{F}(x)$ by parts, yields:

$$\begin{aligned}\int_0^{\gamma^+} u(x)\, d\widehat{F}(x) &= v(x)\, D^1_{\widehat{F}}(x)\Big|_0^{\gamma^+} \\ &\quad - \int_0^{\gamma^+} \frac{dv(x)}{dx} D^1_{\widehat{F}}(x)\, dx.\end{aligned} \tag{71}$$

We thus have:

$$\int_0^a u(x)\, d\widehat{G}(x) - \int_0^a u(x)\, d\widehat{F}(x) = \int_0^{\gamma^+} \frac{dv(x)}{dx} \left[D^1_{\widehat{F}}(x) - D^1_{\widehat{G}}(x) \right] dx. \quad (72)$$

We see that this condition is similar to DSU for $s = 1$ and for $a = \gamma^+$ (see proof of proposition 3). For $s \geq 2$, we can also apply DSU. ■

Proof of proposition 14. The proof of this proposition is trivial. It is simply a matter of noting that, for $s = 1$ and 2 DSPR is equivalent to the DSI* condition when we set $\lambda^+ = \gamma^+$. ■

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Figure 1

First-order inequality dominance

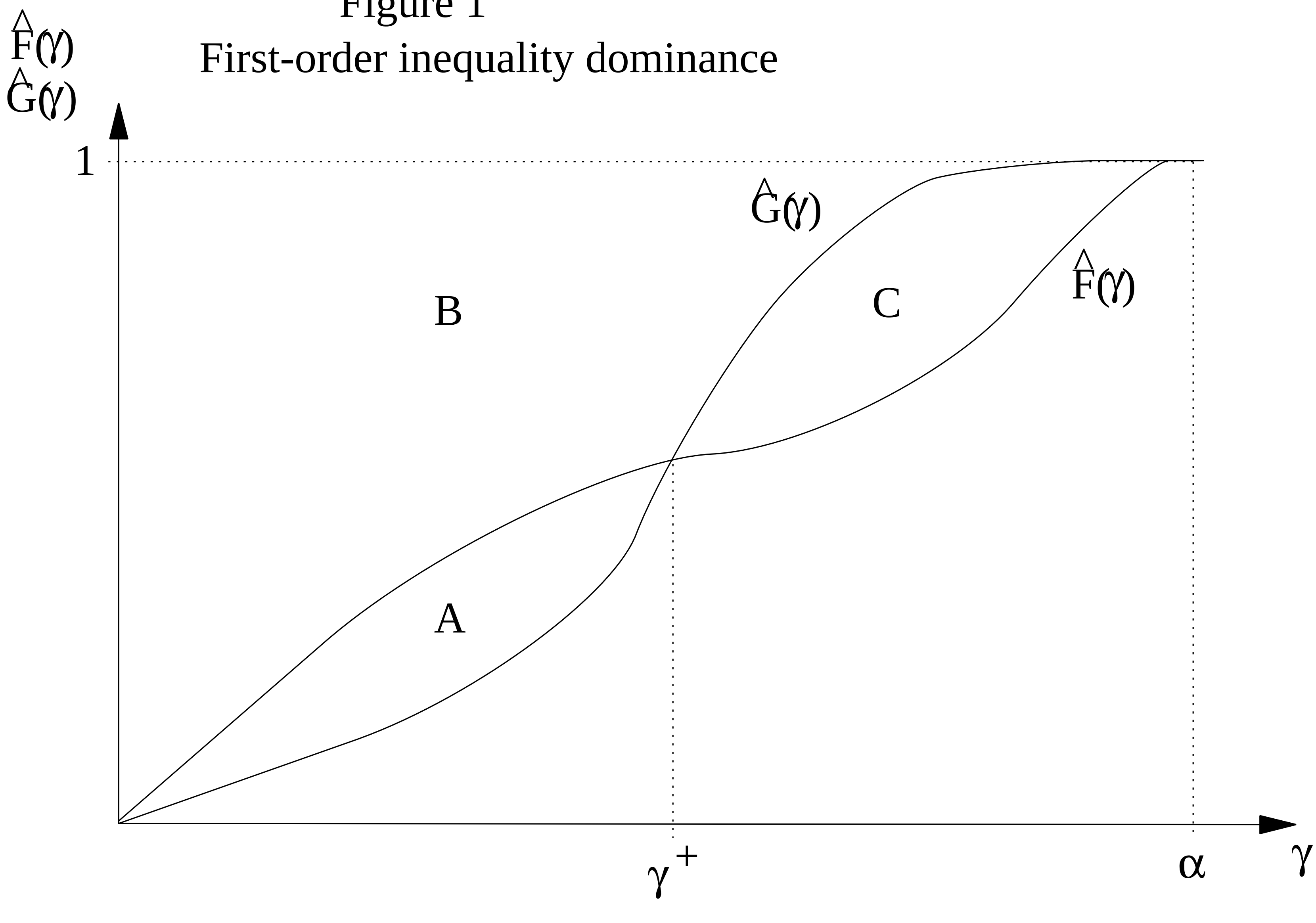
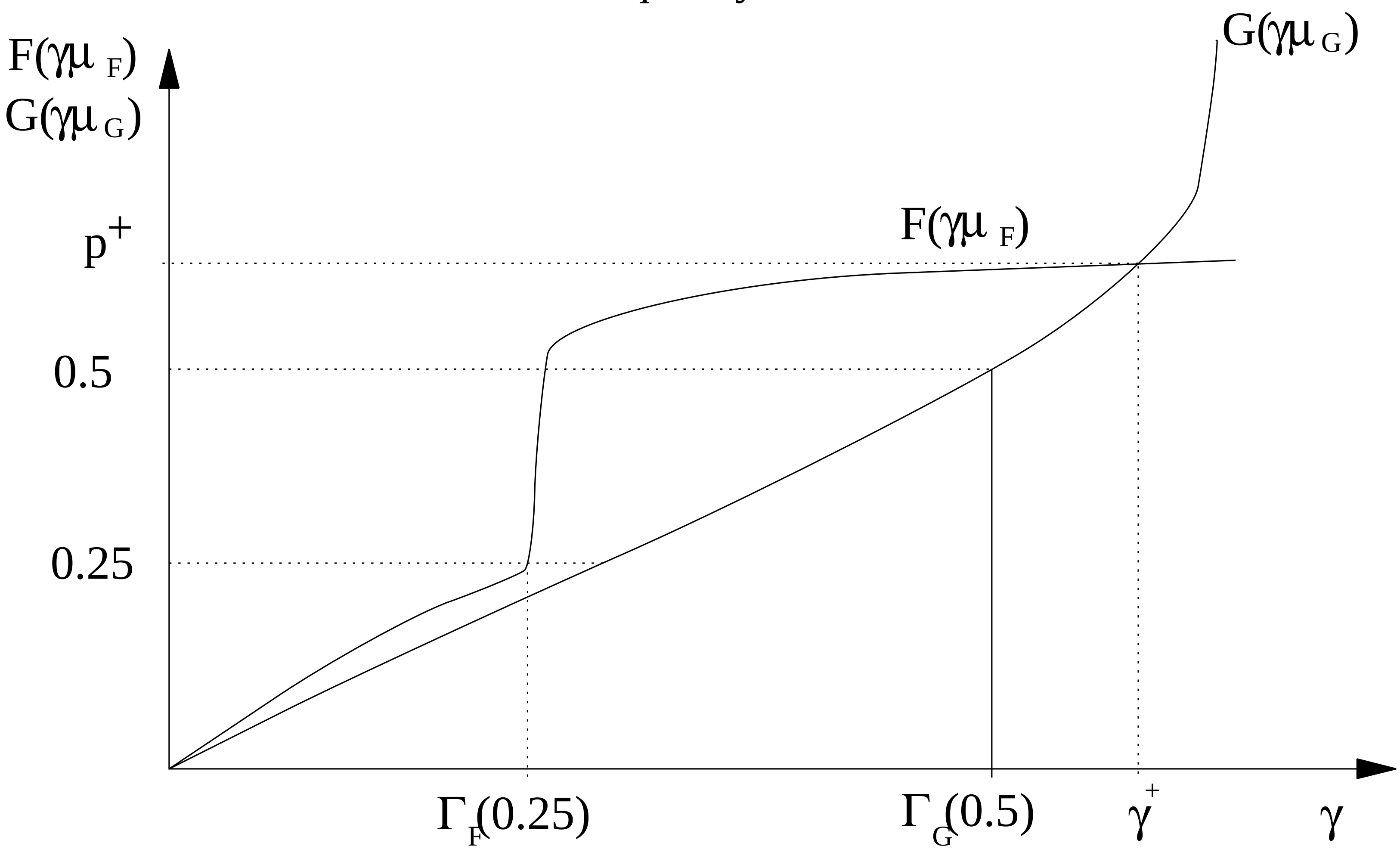


Figure 2
Restricted first-order inequality dominance



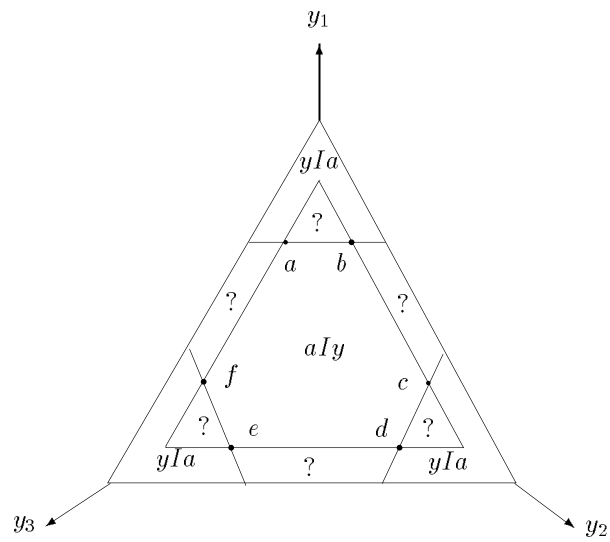


Figure 3: Distribution of 3μ between 3 individuals.

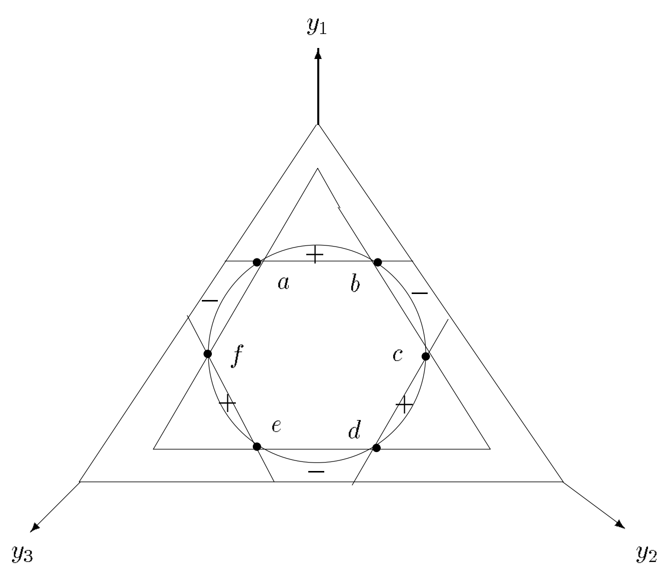


Figure 4: Ordering with Υ^3.

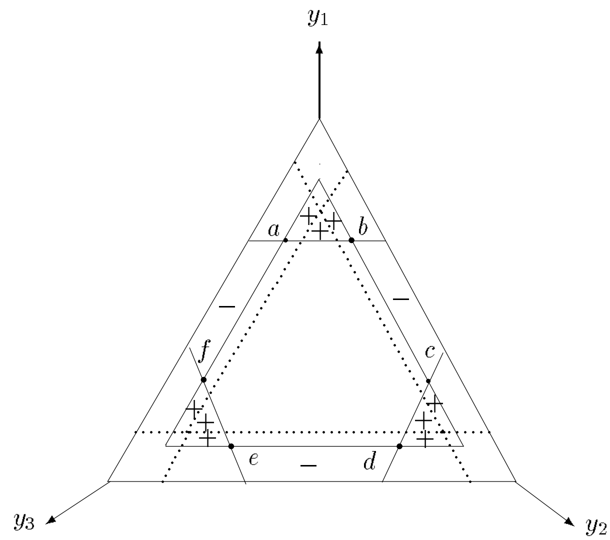


Figure 5: Ordering with Υ^{*2}.

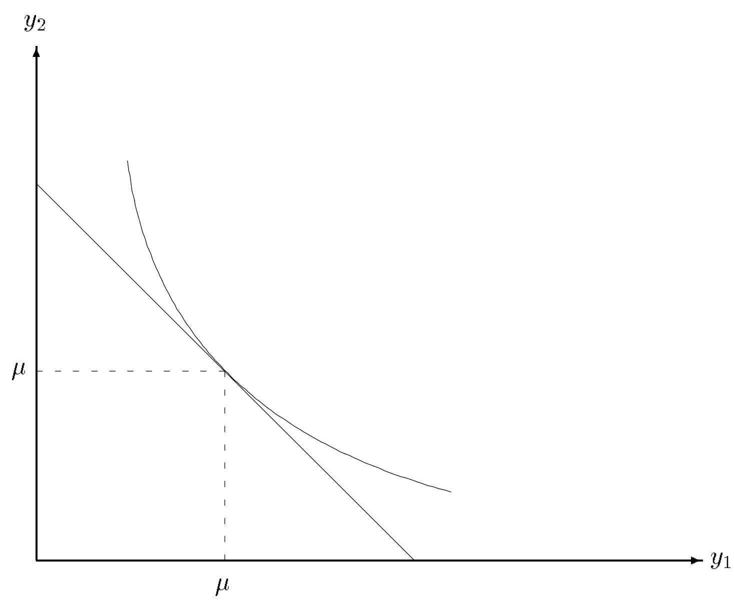


Figure 6: Strictly convex social indifference curves.

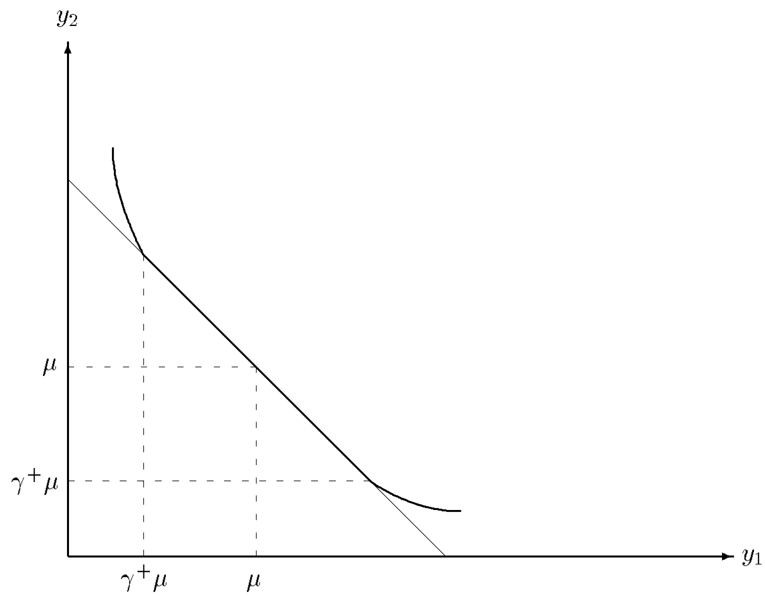


Figure 7: Non-strictly convex social indifference curves.

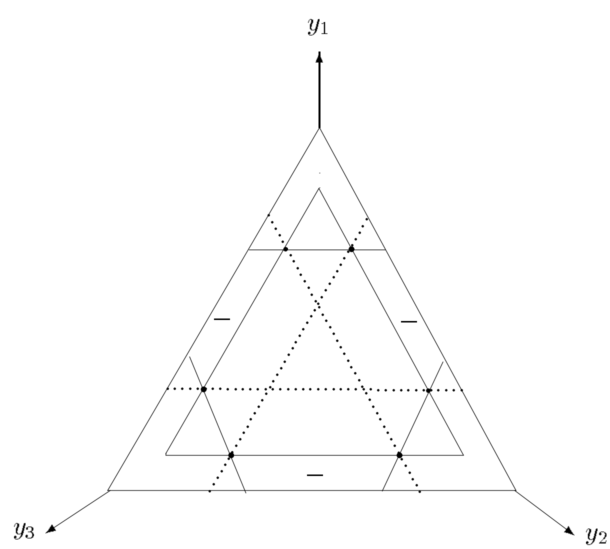


Figure 8: Maximum γ^+ for which the power of ranking a is increased.

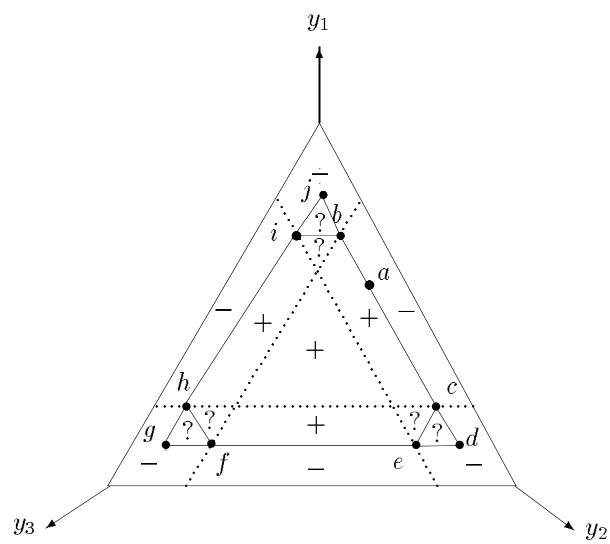


Figure 9: Ordering with Υ^{*1}.

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