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Abstract

The purpose of this paper is to show that the Gini index of equality is: (*i*) subgroup decomposable throughout interpersonal comparisons; (*ii*) decomposable by income source; (*iii*) decomposable both by subgroup and income source; (*iv*) and decomposable in a multidimensional context permitting statistical inference on equality components. These results entail that decision makers must use the Gini index of equality decompositions since they imply necessarily the computation of the Gini index of inequality decompositions. The reverse is not true.

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1 Introduction

In these few last decades the growing literature on the Gini index of inequality measurement has been accompanied by new decomposition techniques. First, the subgroup (subpopulation) decomposition (see e.g. Battacharya and Mahalanobis (1967), Rao (1969), Pyatt (1976), Silber (1989) and Dagum (1997a, 1997b)) enables one to determine the Gini inequality within groups (G_w) and the Gini inequality between groups (G_{gb})[1]. Second, the Gini decomposition by income source (factor component) ascribes a part of the overall inequality to each income constituent (see e.g. Rao (1969), Fei, Ranis and Kuo (1978), Shorrocks (1982)). Recently, Mussard (2004) provides a transparent link between the income source and the subgroup Gini decomposition, the so-called Gini index of inequality multi-decomposition.

The aim of this paper is, first, to point out the use of the Gini index of equality. Indeed, the Gini index of equality constitutes a fundamental step in the computation of many indicators such as the subgroup Sen's index (see Xu-Osberg (2001)) but it is still unused. Why? The computation of the Gini index of inequality (G) yields inevitably the amount of the equality *via* the affine transformation $1 - G$. Nevertheless, when a decomposition is implemented, like a Gini index of inequality decomposition into (G_w) and (G_{gb}), it is not possible to provide, at present time, the equality counterpart of these two elements. Why is this equality decomposition important? Decision makers have two solutions to solve a problem of an increasing inequality: either reduce the amount of inequality or increase the amount of equality. Our approach focuses on the second point, which can also help to solve problems relying on equalization of opportunities.

This paper shows that it is possible to provide three sorts of decomposition: a Gini index of equality decomposition by subgroup, a Gini index of equality decomposition by factor component, and a Gini index of equality multi-decomposition. The main purpose of the article underlies the Gini coefficient of equality multi-decomposition, that is, a breakdown both by source of income and by subgroup. This methodology enables one to compute the couples "sources/within-group" and "sources/between-group" that tend to increase the overall equality (and inequality). This allows many policies of redistribution that focuses on these couples of equality to be contemplated. Furthermore, the Gini index of equality multi-decomposition technique can appear as a more complete approach than the Gini index of inequality multi-decomposition. Indeed, the Gini index of equality is said to be "dual" in the sense that it includes all the Gini index of inequality decompositions (the reverse is not true). Afterwards, as the multi-decomposition yields only an intersection between the dimensions of sources and subgroups, we discuss the possibility to involve our technique in a multidimensional background.

[1]The literature distinguishes the between-group inequality in mean (G_{nb}) and the intensity of transvariation (G_t), i.e., the between-group inequality of overlap (see Dagum (1959, 1960, 1961, 1997a))

The paper is attacked as follows. Section 2 is devoted to the Gini index of equality specification. Section 3 and 4 expose, respectively, the Gini index of equality decomposition by subgroup and by income source. Section 5 and 6 present, respectively, the Gini index of equality multi-decomposition and its extension in a multidimensional context. We conclude in Section 7.

2 The Gini index of equality

2.1 Specification

The Gini index of inequality and the Gini index of equality are related in an explicit way. More specifically, let $W(x) := \phi\overline{W}(x)$ be a homothetic (ordinal) social welfare function with ϕ being an increasing function and $\overline{W}$ being a linearly homogeneous function and x an income distribution. The so-called equally-distributed-equivalent-income (ΞDEI), ξ, is an income that makes $\overline{W}(x) = \overline{W}(\xi)$. That is, the ΞDEI is the income that generates the same level of social welfare when the ΞDEI is distributed to everyone to replace his or her actual income. Since $\overline{W}$ is positively linearly homogeneous, the ΞDEI is given by $\xi = \overline{W}(x)/\overline{W}(1\mathbf{e})$, denoting $\mathbf{e}$ by the n-tuple of 1's, and n the size of the income distribution. Let μ be the mean income and x_i the income of the i-th individual. If the Gini social welfare function,

$$\Xi_G(X) = \frac{1}{n^2}\sum_{i=1}^{n}(2n - 2i + 1)x_i, \tag{2.1}$$

is adopted, then the Gini index of inequality is given by

$$G = 1 - \frac{\Xi_G(X)}{\mu} \tag{2.2}$$

while the Gini index of equality is defined as

$$G_e = \frac{\Xi_G(X)}{\mu}. \tag{2.3}$$

Thus, we have the Gini index of equality, G_e, defined as $G_e := 1 - G$, where G can be expressed as:

$$G = \frac{\sum_{i=1}^{n}\sum_{r=1}^{n}|x_i - x_r|}{2n^2\mu}, \tag{2.4}$$

and where x_r stands for the r-th individual's income. The denominator of G can be formulated as:

$$2n^2\mu = \sum_{i=1}^{n}\sum_{r=1}^{n}(x_i + x_r). \tag{2.5}$$

Hence, equation (2.3) can be rewritten as:

$$G_e = 1 - G = \frac{\sum_{i=1}^{n}\sum_{r=1}^{n}(x_i + x_r) - \sum_{i=1}^{n}\sum_{r=1}^{n}|x_i - x_r|}{2n^2\mu}. \qquad (2.6)$$

Although the Gini index of equality is not as well-known as the Gini index of inequality, it has desirable properties and its mathematical structure offers thorough decompositions. However, before discussing decomposition, it is crucial to know if the Gini index of equality satisfies the standard axioms of this literature.

2.2 Axiomatic Properties

Let I be the generic term for both the measures of inequality and equality, with D, D_j, D_h three income distributions. Let $\mathbb{D}$ be the set of all income distributions $\mathbb{D} = \bigcup_{n\in\mathbb{N}} \mathbb{R}^n$, where $\mathbb{D}_+$, $\mathbb{R}^n$, and $\mathbb{R}_+$ stand for the non-negative part of $\mathbb{D}$, the Euclidean n-space, and its one-dimensional-positive counterpart, respectively. The standard axioms for inequality measurement are the following.

First, I must be sensitive to transfers between individuals.

Axiom 2.1 ***Pigou-Dalton Principle*** (PD). *An index of inequality $I(.) : \mathbb{D}_+ \longrightarrow \mathbb{R}_+$, is said to be consistent with PD if a distribution D_j is issued from D_h by a finite sequence of progressive transfers (transfers from a rich person to a poor person):*

$$I(D_j) < I(D_h). \qquad (2.7)$$

Next, we require that I enables income distributions with different sizes to be compared.

Axiom 2.2 ***Population Principle*** (PP). *The index $I(.) : \mathbb{D}_+ \longrightarrow \mathbb{R}_+$, is said to be consistent with PP if a distribution $D_j^{(t)}$ generated from concatenating D_j t times satisfies the following expression:*

$$I(D_j^{(t)}) = I(D_j), \forall j \geq 2. \qquad (2.8)$$

The intensity of the measure I does not depend on the identities of the individuals.

Axiom 2.3 ***Symmetry*** (SM). *The index $I(.) : \mathbb{D}_+ \longrightarrow \mathbb{R}_+$, is said to be consistent with SM if a distribution D_j generated from permuting the elements of another distribution D_h does not alter the value of the index for D_h:*

$$I(D_j) = I(D_h). \qquad (2.9)$$

The I index is normalized.

Axiom 2.4 ***Normalization*** (NM). *The index $I(.) : \mathbb{D}_+ \longrightarrow \mathbb{R}_+$, is said to be consistent with NM if it takes values in the close interval* $[0, 1]$.

The measure I is homogeneous of degree zero.

Axiom 2.5 ***Ratio Scale Invariance*** *(RS). The index $I(.) : \mathbb{D}_+ \longrightarrow \mathbb{R}_+$ is ratio scale invariant or homogeneous of degree zero if it satisfies:*

$$I(\lambda D) = I(D), \forall \lambda > 0. \tag{2.10}$$

Theorem 2.6 *The Gini index of inequality satisfies PD, PP, SM, NM, and RS.*

Proof: See Ebert (1988) and Deutsch and Silber (1999).

Corollary 2.7 *The Gini index of equality, $G_e = 1 - G$, also satisfies PD, PP, SM, NM, and RS.*

Proof: Up to a change of sign, as G belongs to [0,1] $\Longrightarrow G_e : \mathbb{D}_+ \longrightarrow [0,1]$, where G_e equals 1 when the perfect equality in incomes is reached and tends toward 0 when the income distribution is unequal. It is obvious that G_e respects RS since the Gini inequality index satisfies RS: $G_e(\lambda D) = 1 - G(\lambda D) = 1 - G(D) = G_e(D)$. As G_e is the opposite index of G, when D_j is issued from D_h by a finite sequence of progressive transfers, we have: $G(D_j) < G(D_h) \Longleftrightarrow 1 - G_e(D_j) < 1 - G_e(D_h) \Longleftrightarrow G_e(D_j) > G_e(D_h)$. Then, G_e satisfies PD in the sense that the equality index increases after a sequence of progressive transfers (and conversely). Given that $G_e(D^{(t)}) = 1 - G(D^{(t)}) = 1 - G(D) = G_e(D)$, G_e respects PP. Let Π be a n by n permutation matrix, we obtain: $G_e(\Pi D) = 1 - G(\Pi D) = 1 - G(D) = G_e(D)$, since the Gini index of inequality satisfies SM. □

We have shown that the particular affine transformation of G into $1 - G$ implies the respect of the same axioms. However, anything proves that the different Gini inequality decompositions imply the same decomposition structure for the Gini equality ratio.

3 The Subgroup decomposition of the Gini index of equality

3.1 New design of subgroup decomposition

The entropy and the Gini index of inequality satisfy suitable properties. The domain of subgroup decompositions is often characterized by the generalized entropy inequality indices. They are issued from the second law of thermodynamics that gauges a system disorder. Following Shorrocks (1984, 1988) these measures are more attractive since there is a close interrelation with the subgroup consistency property. Let us remember. Consider a population P with n income units x_i ($\forall i = 1, \ldots, n$) of mean income μ. The population is partitioned into K subgroups P_k, with mean income μ_k ($\forall k = 1, \ldots, K$). Let s_k and p_k be the income share and population share related to the k-th group, respectively: $s_k = \frac{n_k \mu_k}{n\mu}$; $p_k = \frac{n_k}{n}$.

Definition 3.1 ***Subgroup Consistency*** (SC). *Suppose a change in incomes of group P_k, ceteris paribus, group size and mean income are fixed. If the inequality goes up (down) in group P_k, then the overall inequality increases (decreases).*

In contrast to this, if an inequality measure is not subgroup consistent, we can have the particular situation where the inequality rises in every subgroup while the overall inequality declines. The SC property is rational but the logic is flaw. Why does the whole inequality increase when the inequality goes up within one or several groups? The subpopulation decomposition is: $I = I_w + I_b$, where I is the overall inequality measure, I_w the within-group component that measures the inequalities within groups, and I_b the between-group component that measures the inequalities in mean between groups. Therefore, a change in income of group P_k, *ceteris paribus*, does not modify I_b since it only depends on population shares and income shares.

Following Dagum (1997a) this property produces between-group measures of inequality that can not represent a valid specification of a statistical measure, since I_b is very close to the one-way variance analysis ($ANOVA$) for which: (i) the subgroups have equal variances; (ii) the observations are statistically independent; and (iii) the distributions are equally distributed. Let us suggest another decomposition property.

Definition 3.2 ***Decomposition Based on Interpersonal Comparisons***. *An index of (in)equality satisfies the property of decomposability based on interpersonal comparisons if the within-group index involves the interpersonal comparisons in incomes within each subgroup and if the between-group index includes the interpersonal comparisons in incomes between each and every pair of subpopulations.*

This definition can be characterized by many family of indices such as Kolm's (1999) pair-based measures of inequality $I(x) := F[\phi(x_i, x_r)]$, where $F(0) = 0, F > 0$, ϕ is symmetrical, $\phi(\xi, \xi) = 0$, and $\phi > 0$ in other cases. An interesting case is $F := f(\sum_{i<j} g[\phi(x_i, x_r)])$, where f and g are increasing and where $f(0) = g(0) = 0$. This definition means that the subgroup decomposition relies on income pairs, for instance income differences, which is reasonable in the case of inequality measures. Therefore, up to the transformation F, gathering the income differences $f(x_i, x_r) := |x_i - x_r|$ within groups and between groups, we bring out within-group and between-group components of income inequality indices, which avoid the $ANOVA$ critical insights (see e.g. Dagum (1997a) and Berrebi and Silber (1987) for the Gini index of inequality).

3.2 Subgroup decomposition

Following the definition of the decomposition based on interpersonal comparisons, it is possible to implement the subgroup decomposition of the Gini index of equality.

Lemma 3.3 *Given a partition of the whole population P in K subgroups, it is possible to compute the Gini index of equality for each subgroup.*

Proof: As shown previously, the Gini index of equality for the whole population is:

$$G_e = 1 - G = \frac{\sum_{i=1}^{n}\sum_{r=1}^{n}(x_i + x_r) - \sum_{i=1}^{n}\sum_{r=1}^{n}|x_i - x_r|}{2n^2\mu}. \tag{3.1}$$

Therefore, it is possible to define the Gini index of equality for the k-th subgroup, in which there are n_k incomes $(x_{1k}, x_{2k}, \ldots, x_{nkk})$ as:

$$G_{ekk} = \frac{\sum_{i=1}^{n_k}\sum_{r=1}^{n_k}(x_{ik} + x_{rk}) - \sum_{i=1}^{n_k}\sum_{r=1}^{n_k}|x_{ik} - x_{rk}|}{2n_k^2\mu_k} \tag{3.2}$$

or

$$G_{ekk} = \frac{\sum_{i=1}^{n_k}\sum_{r=1}^{n_k}\left[(x_{ik} + x_{rk}) - |x_{ik} - x_{rk}|\right]}{2n_k^2\mu_k}, \forall k = 1, \ldots, K. \tag{3.3}$$

These indices are similar to the Gini index of inequality in that a perfect equalitarian distribution within the k-th subgroup yields:

$$\sum_{i=1}^{n_k}\sum_{r=1}^{n_k}|x_{ik} - x_{rk}| = 0, \sum_{i=1}^{n_k}\sum_{r=1}^{n_k}(x_{ik} + x_{rk}) = 2n_k^2\mu_k$$
$$\Longrightarrow G_{ekk} = 1. \;\square \tag{3.4}$$

Lemma 3.4 *Given a partition of the whole population P in K subgroups, it is possible to compute the Gini index of equality between each pair of subgroups P_k and P_h.*

Proof: Given the subgroups P_k and P_h, it is possible to define the Gini index of equality between the k-th and the h-th subgroups as:

$$G_{ekh} = \frac{\sum_{i=1}^{n_k}\sum_{r=1}^{n_h}\left[(x_{ik} + x_{rh}) - |x_{ik} - x_{rh}|\right]}{n_k n_h(\mu_k + \mu_h)}. \tag{3.5}$$

With this definition, we can measure the index G_{ekh} for all h, $k = 1, \ldots, K$ $(h \neq k)$. Given a perfect equality between two distributions, even if their size are different [such as $P_k = (2, 2, 2)$ and $P_h = (2, 2, 2, 2)$], then the Gini index of equality between groups P_k and P_h is valued to be 1, that is:

$$\sum_{i=1}^{n_k}\sum_{r=1}^{n_h}|x_{ik} - x_{rh}| = 0, \sum_{i=1}^{n_k}\sum_{r=1}^{n_h}(x_{ik} + x_{rh}) = n_k n_h(\mu_k + \mu_h)$$
$$\Longrightarrow G_{ekh} = 1. \tag{3.6}$$

Note that if $h = k$, the Gini index of equality between two subgroups G_{ekh} (3.5) yields the Gini index associated with one subgroup (3.3). $\square$

Theorem 3.5 *The total Gini ratio of equality of a population P partitioned in K subgroups of size n_k, can be decomposed into within-group and between-group elements. These elements are respectively weighted average of Gini indices of equality associated with one group and Gini indices of equality between each and every pair of subgroups.*

Proof: The Gini index of equality of the population P is:

$$G_e = \frac{\sum_{i=1}^{n} \sum_{r=1}^{n} [(x_i + x_r) - |x_i - x_r|]}{(2n^2 \mu)}. \tag{3.7}$$

Define the indicator $t_{ir} = [(x_i + x_r) - |x_i - x_r|]$, which increases its value as x_i and x_r get to be closer to each other and reaches the maximum for given i and r when $x_i = x_r$. Therefore, to assess the degree of equality within the k-th subgroup, we must find the sum of terms t_{ir}, for all i, r which belong to the k-th subgroup. This degree of equality can be obtained by gathering the following income pairs within every k-th group: $(x_{ik} + x_{rk})$ and $|x_{ik} + x_{rk}|$. The first term represents the pure equality between the i-th and the r-th individual whereas the second one characterizes the inequality between them:[2]

$$G_{ew} = \frac{\sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{r=1}^{n_k} [(x_{ik} + x_{rk}) - |x_{ik} - x_{rk}|]}{2n^2 \mu}. \tag{3.8}$$

Remember that p_k is the proportion of individuals within the k-th subgroup, and that s_k is the income proportion of the k-th subgroup:

$$p_k = \frac{n_k}{n}, s_k = \frac{n_k \mu_k}{n \mu}. \tag{3.9}$$

Using (3.9) and Lemma (3.3), we rewrite the within-group Gini index of equality as:

$$G_{ew} = \sum_{k=1}^{K} p_k s_k G_{ekk}. \tag{3.10}$$

On the other hand, in order to evaluate the degree of equality between subgroups, we gauge the differences $t_{ir} = [(x_i + x_r) - |x_i - x_r|]$ between the groups, where x_i belongs to P_k and x_r to P_h. When $x_i = x_r$, t_{ir} reaches its maximum for given i and r. We must then gather all t_{ir}'s that prevail between each and every x's pair from the two subgroups. This yields the gross between-group Gini index of equality:

$$G_{egb} = \frac{\sum_{k=2}^{K} \sum_{h=1}^{k-1} \sum_{i=1}^{n_k} \sum_{r=1}^{n_h} [(x_{ik} + x_{rh}) - |x_{ik} - x_{rh}|]}{2n^2 \mu}. \tag{3.11}$$

[2]It would be desirable to choose $max\{x_{ik}, x_{rk}\}$ as the income of perfect equality between the i-th and r-th individuals. This would entail the computation of the Gini mean ratio index of equality, see Mussard (2004).

This index measures the intensity of equality between all pairs of subgroups although it is an unconventional between-group measure. Typically, as shown previously with the SC property, between-group indices measure the relationship among subgroup mean incomes. However, the gross between-group Gini index of equality defined here provides richer information because this "gross" between-group element (Dagum's (1997a) definition) reflects a weighted average of the Gini indices of equality between each and every pair of subgroups. Indeed, Lemma (3.4) entails:

$$G_{egb} = \sum_{k=2}^{K} \sum_{h=1}^{k-1} G_{ekh}(p_k s_h + p_h s_k). \tag{3.12}$$

Finally, the overall Gini index of equality is the sum of the within-group Gini index of equality [G_{ew}] and the gross between-group Gini index of equality [G_{egb}]:

$$G_e = \underbrace{\sum_{k=1}^{K} p_k s_k G_{ekk}}_{G_{ew}} + \underbrace{\sum_{k=2}^{K} \sum_{h=1}^{k-1} G_{ekh}(p_k s_h + p_h s_k)}_{G_{egb}}. \; \Box \tag{3.13}$$

Remark 3.6 *Following Dagum (1997a), the within-group Gini index of inequality is:*

$$G_w = \sum_{k=1}^{K} p_k s_k G_{kk}, \tag{3.14}$$

where G_{kk} is the Gini index of inequality associated with the k-th subgroup. As can be seen in equation (3.10)*, there is a clear connection between G_{ew} and G_w.*

Remark 3.7 *The gross between-group Gini index of inequality, see Dagum (1997a), is:*

$$G_{gb} = \sum_{k=2}^{K} \sum_{h=1}^{k-1} G_{kh}(p_k s_h + p_h s_k), \tag{3.15}$$

which has a close connection with the gross between-group Gini index of equality given in equation (3.12)*.*

Corollary 3.8 *The within-group Gini index of equality G_{ew} entails the computation of the within-group Gini index of inequality.*

Proof: The within-group Gini index of equality can be rewritten as:

$$\begin{aligned} G_{ew} &= \frac{\sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{r=1}^{n_k} [(x_{ik} + x_{rk}) - |x_{ik} - x_{rk}|]}{2n^2\mu} \\ &= \frac{\sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{r=1}^{n_k} (x_{ik} + x_{rk})}{2n^2\mu} - G_w, \end{aligned} \tag{3.16}$$

where,

$$G_w = \frac{\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{r=1}^{n_k}|x_{ik}-x_{rk}|}{2n^2\mu}, \tag{3.17}$$

is the within-group Gini index of equality (see Dagum (1997a)). The gross between-group Gini index of equality can also be reformulated as:

$$\begin{aligned} G_{egb} &= \frac{\sum_{k=2}^{K}\sum_{h=1}^{k-1}\sum_{i=1}^{n_k}\sum_{r=1}^{n_h}[(x_{ik}+x_{rh})-|x_{ik}-x_{rh}|]}{2n^2\mu} \\ &= \frac{\sum_{k=2}^{K}\sum_{h=1}^{k-1}\sum_{i=1}^{n_k}\sum_{r=1}^{n_h}(x_{ik}+x_{rh})}{2n^2\mu} - G_{gb}, \end{aligned} \tag{3.18}$$

where,

$$G_{egb} = \frac{\sum_{k=2}^{K}\sum_{h=1}^{k-1}\sum_{i=1}^{n_k}\sum_{r=1}^{n_h}|x_{ik}-x_{rh}|}{2n^2\mu}, \tag{3.19}$$

is the gross between-group index of inequality (see Dagum (1997a)). □

This Corollary shows that the Gini index of equality is a dual measure of subgroup decomposition since it yields both the equality and the inequality intensities within groups and between groups, whereas the computation of the Gini index of inequality decomposition only gives a conclusion in terms of inequality. On this basis, the computation of the Gini index of equality decomposition seems to be desirable.

Corollary 3.9 . *If the K distributions are equally distributed ($x_{ik}=\mu, \forall i = 1,\ldots,n_k$ and $\forall k = 1,\ldots,K$), then:*

$$G_{ekk} = G_{ekh} = G_e = G_{ew} + G_{egb} = 1. \tag{3.20}$$

Proof: If the distributions are equally distributed, $G_{ekk} = G_{ekh} = 1$, then:

$$G_{ew} = \sum_{h=1}^{K}\frac{n_k n_k \mu_k}{nn\mu}G_{ekk} = \sum_{h=1}^{K}\frac{n_k^2}{n^2}. \tag{3.21}$$

On the other hand, given that $G_{ekh}=1$ and $\mu=\mu_k=\mu_h$ ($\forall k,h=1,\ldots,K$), we immediately obtain:

$$\begin{aligned} G_{egb} &= \sum_{k=2}^{K}\sum_{h=1}^{k-1}G_{ekh}(p_k s_h + p_h s_k) \\ &= \sum_{k=2}^{K}\sum_{h=1}^{k-1}1\left(\frac{n_k n_h \mu_h}{nn\mu}+\frac{n_k n_h \mu_k}{nn\mu}\right) \\ &= \sum_{k=2}^{K}\sum_{h=1}^{k-1}\frac{2n_k n_h}{n^2}. \end{aligned} \tag{3.22}$$

Thus:

$$G_e = G_{ew} + G_{egb} = \sum_{k=1}^{K} \frac{n_k^2}{n^2} + \sum_{k=2}^{K} \sum_{h=1}^{k-1} (\frac{2n_k n_h}{n^2}) = 1. \square \tag{3.23}$$

Corollary 3.10 . *If the* K *subgroups are* K *replications* ($i.i.d.$ *distributions*), *then* $\mu_k = \mu_h = \mu$ $(\forall k, h = 1, \ldots, K)$ *and:*

$$G_e = G_{ekk} = G_{ekh} \ (\forall k, h = 1, \ldots, K); \tag{3.24}$$

$$\Longrightarrow G_{ew} = \frac{1}{K} G_e \ ; \ G_{egb} = \frac{K-1}{K} G_e. \tag{3.25}$$

Proof: Let D_k be the distribution of the k-th subgroup and $D_k^{(n)}$ the distribution generated from replicating D_k n times. Given that the overall income distribution D is the concatenation of the D_k's and given that G_e satisfies the population principle, it must be the case that:

$$G_e(D_k) = G_e\left(D_k^{(n)}\right) = G_e(D). \tag{3.26}$$

Then, $D_k = D_h$ produces $G_{ekk} = G_{ekh}$. Indeed, as the Gini equality coefficient between two subgroups is a generalization of the Gini equality coefficient associated with one group, then $D_k = D_h \Longrightarrow n_k = n_h, \mu_k = \mu_h$, thus:

$$\begin{aligned} G_{ekh} &= \frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_h} [(x_{ik} + x_{rh}) - |x_{ik} - x_{rh}|]}{n_k n_h (\mu_k \mu_h)} \\ &= \frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_k} [(x_{ik} + x_{rk}) - |x_{ik} - x_{rk}|]}{n_k n_k (\mu_k \mu_k)} \\ &= G_{ekk} \end{aligned} \tag{3.27}$$

For $i.i.d.$ distributions we have, for each subgroup P_k, a size $n_k = n/K$. The within-group Gini index of equality is:

$$G_{ew} = \sum_{k=1}^{K} \frac{n_k^2}{n^2} G_{ekk} = \sum_{k=1}^{K} \frac{1}{K^2} G_{ekk} = \frac{G_{ekk}}{K} = \frac{G_e}{K}. \tag{3.28}$$

The gross between-group Gini index of equality is:

$$\begin{aligned} G_{egb} &= \sum_{k=2}^{K} \sum_{h=1}^{k-1} G_{ekh}(p_k s_h + p_h s_k) = \sum_{k=2}^{K} \sum_{h=1}^{k-1} \left(\frac{n_k n_h \mu_h}{n n \mu} + \frac{n_k n_h \mu_k}{n n \mu} \right) G_e \\ &= \sum_{k=2}^{K} \sum_{h=1}^{k-1} \frac{2 n_k n_h}{n^2} G_e = \sum_{k=2}^{K} \sum_{h=1}^{k-1} \frac{2}{K^2} G_e \\ &= 2\mathbf{C}_K^2 \frac{G_e}{K^2} = (K^2 - K) \frac{1}{K^2} G_e = \left(1 - \frac{1}{K}\right) G_e, \end{aligned} \tag{3.29}$$

where $\mathbf{C}_K^2$ is the number of pairwise combinations (without replacement) within a set of K elements. $\square$

4 The Source decomposition of the Gini index of equality

The source decomposition allows the contribution of each source (factor) of income to the overall equality to be determined. While the denominator of G_e plays the role of normalization, the numerator still possesses an additive configuration. Suppose that each individual's income is composed of M sources such as labour income, capital income, transfers, child support benefits, etc. $(m = 1, \ldots, M)$:

$$x_i = \sum_{m=1}^{M} x_i^m. \tag{4.1}$$

Theorem 4.1 *The Gini index of equality is decomposable by income source so that it enables one to determine the contribution of each source constituent to the overall equality.*

Proof: Given that individuals's incomes are disaggregated as (4.1), it is possible to rewrite G_e as:

$$G_e = \frac{\sum_{i=1}^{n} \sum_{r=1}^{n} \left[\sum_{m=1}^{M} x_i^m + \sum_{m=1}^{M} x_r^m - | \sum_{m=1}^{M} x_i^m - \sum_{m=1}^{M} x_r^m | \right]}{2\mu n^2}. \tag{4.2}$$

Given $|a - b| = a + b - 2min\{a, b\}$, the overall Gini index of equality is:

$$\begin{aligned} G_e =& \frac{\sum_{i=1}^{n} \sum_{r=1}^{n} [\sum_{m=1}^{M} x_i^m + \sum_{m=1}^{M} x_r^m - \sum_{m=1}^{M} x_i^m - \sum_{m=1}^{M} x_r^m}{2\mu n^2} \\ &+ \frac{2min\{\sum_{m=1}^{M} x_i^m, \sum_{m=1}^{M} x_r^m\}]}{2\mu n^2}. \end{aligned} \tag{4.3}$$

Let $\sum_{m=1}^{M} x_{ir}^{*m}$ be an operator which selects the minimum of x_i and x_r and then decomposes the minimum income (either x_i or x_r) in M sources. Hence, the Gini equality ratio becomes:

$$\begin{aligned} G_e =& \frac{\sum_{i=1}^{n} \sum_{r=1}^{n} [\sum_{m=1}^{M} x_i^m + \sum_{m=1}^{M} x_r^m - \sum_{m=1}^{M} x_i^m - \sum_{m=1}^{M} x_r^m}{2\mu n^2} \\ &+ \frac{2 \sum_{m=1}^{M} x_{ir}^{*m}]}{2\mu n^2}. \end{aligned} \tag{4.4}$$

Given that the summation operator is commutative and associative, we have:

$$\begin{aligned} G_e &= \sum_{m=1}^{M} \frac{\sum_{i=1}^{n} \sum_{r=1}^{n} [x_i^m + x_r^m - x_i^m - x_r^m + 2x_{ir}^{*m}]}{2\mu n^2} \\ &= \sum_{m=1}^{M} \frac{\sum_{i=1}^{n} \sum_{r=1}^{n} 2x_{ir}^{*m}}{2\mu n^2}. \end{aligned} \tag{4.5}$$

Consequently, the sum of each source contribution leads to the overall equality,

$$G_e = \sum_{m=1}^{M} G_e^m, \tag{4.6}$$

where the contribution of the m-th source to the overall Gini index of equality is expressed as:

$$G_e^m := \frac{\sum_{i=1}^n \sum_{r=1}^n [2x_{ir}^{*m}]}{2\mu n^2}. \square \tag{4.7}$$

As can be seen above, the source decomposition is exact in the sense that each contribution corresponds to a factor component. As pointed out by Shorrocks (1982), many income source decompositions yield a path dependence between the aggregated income and its constituents (see Rao (1969) and Fei, Ranis, Kuo (1978)), in particular on the fact that:

$$2\sum_{m=1}^{M} x_{ir}^{*m} := 2min\{x_i, x_r\}. \tag{4.8}$$

If $x_i < x_r$, then x_i determines the level of the source contribution. On the other hand, when equal incomes are observed, $\sum_{m=1}^M x_i^m = \sum_{m=1}^M x_r^m$, there is a problem to determine the minimum between x_i and x_r. As the i-th and the r-th individuals are compared two times, one can equilibrate the decomposition by taking $\sum_{m=1}^M x_i^m$ in the first interpersonal comparison and $\sum_{m=1}^M x_r^m$ in the second one. That is, we impose this condition to make sure the decomposition is consistent with the aggregate Gini index of equality.

Corollary 4.2 *The Gini index of equality involves as a particular case the specification of the income source Gini index of inequality decomposition.*

Proof: Following (3.1), we have:

$$G_e = \frac{\sum_{i=1}^n \sum_{r=1}^n (\sum_{m=1}^M x_i^m + \sum_{m=1}^M x_r^m)}{2\mu n^2} - \sum_{m=1}^{M} G^m, \tag{4.9}$$

where

$$G^m = \frac{\sum_{i=1}^n \sum_{r=1}^n (x_i^m + x_r^m - 2x_{ir}^{*m})}{2\mu n^2} \tag{4.10}$$

is the contribution of the m-th source to the overall Gini index of inequality (see Mussard (2004)). $\square$

5 The Multi-Decomposition of the Gini index of equality

Mussard (2004, 2005) demonstrates that the Gini index of inequality is simultaneously decomposable by subgroup and income source. It permits all the missing values "$\times$" of the following table to be computed.

Table 1: Structure of the multi-decomposition

Sources → Groups ↓	Source 1	...	Source M	Total
Inequalities in group 1	$\times$	...	$\times$	(.)%
⋮	$\times$	...	$\times$	⋮
Inequalities in group K	$\times$	...	$\times$	(.)%
Inequalities between 1 / 2	$\times$	...	$\times$	(.)%
⋮	$\times$	...	$\times$	⋮
Inequalities between $K-1$ / K	$\times$	...	$\times$	(.)%
Total	(.)%	...	(.)%	100%

In this section, we show that the Gini index of equality yields the same two-by-two table in terms of equality and furthermore produces the couples of the Gini index of inequality multi-decomposition. The Gini equality coefficients (G_{ekk}, G_{ekh}) have the same structure as those of the overall Gini index of equality G_e. Consequently, we can decompose these indices by income source.

Lemma 5.1 *The Gini index associated with one subgroup G_{ekk} and with two subgroups G_{ekh} are source decomposable.*

Proof: Following theorem (4.1), we have:

$$G_{ekk} = \sum_{m=1}^{M} \frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_k} 2x_{irk}^{*m}}{2\mu_k n_k^2} = \sum_{m=1}^{M} G_{ekk}^{m}, \tag{5.1}$$

and

$$G_{ekh} = \sum_{m=1}^{M} \frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_h} 2x_{irkh}^{*m}}{n_k n_h(\mu_k + \mu_h)} = \sum_{m=1}^{M} G_{ekh}^{m}. \tag{5.2}$$

Hence, it is possible to implement the contribution of the m-th factor component ($m = 1, \ldots, M$) to the income equality within the subpopulation P_k and between the subpopulations P_k and P_h. $\square$

Given that the above Gini indices of equality are decomposable by income source, we introduce them into the subgroup decomposition to contemplate the equality multi-decomposition.

Theorem 5.2 *The total Gini index of equality of a population P of size n partitioned into K groups with incomes separated into M factor components is multi-decomposable in the sense of permitting "within-group/sources" and "between-group/sources" kinds of decomposition, which together account for the aggregate degree of equality exactly.*

Proof: Following Lemma (5.1), substituting equations (5.1) and (5.2) into the subgroup decomposition (3.13), we obtain the multi-decomposition of the Gini index of equality:

$$\begin{aligned} G_e &= \sum_{k=1}^{K} p_k s_k \sum_{m=1}^{M} G_{ekk}^m + \sum_{k=2}^{K} \sum_{h=1}^{k-1} (p_k s_h + p_h s_k) \sum_{m=1}^{M} G_{ekk}^m \\ &= \underbrace{\sum_{k=1}^{K} \sum_{m=1}^{M} p_k s_k G_{ekk}^m}_{G_{ew}} + \underbrace{\sum_{k=2}^{K} \sum_{h=1}^{k-1} \sum_{m=1}^{M} (p_k s_h + p_h s_k) G_{ekh}^m}_{G_{egb}} . \end{aligned} \tag{5.3}$$

We must then demonstrate that this multi-decomposition is exact, i.e., a decomposition that simultaneously breaks down all the elements (except the denominator) by source and by subgroup without redundant terms. After many simple algebraic manipulations, the within-group Gini index of equality is seen to be exactly multi-decomposable:

$$G_{ew} = \sum_{m=1}^{M} \frac{\sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{r=1}^{n_k} 2x_{irk}^{*m}}{2\mu n^2}. \tag{5.4}$$

The gross between-group component is also exactly multi-decomposable. More specifically, we how that:

$$G_{egb} = 2 \sum_{k=2}^{K} \sum_{h=1}^{k-1} \sum_{m=1}^{M} \frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_h} 2x_{irkh}^{*m}}{2\mu n^2}. \tag{5.5}$$

Because of equations (5.4) and (5.5), the Gini index of equality is perfectly multi-decomposable as given below:

$$G_e = \underbrace{\sum_{k=1}^{K} \sum_{m=1}^{M} \frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_k} 2x_{irk}^{*m}}{2\mu n^2}}_{G_{ew}} + \underbrace{2 \sum_{k=2}^{K} \sum_{h=1}^{k-1} \sum_{m=1}^{M} \frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_h} 2x_{irkh}^{*m}}{2\mu n^2}}_{G_{egb}} . \; \square \tag{5.6}$$

This theorem signifies that the overall equality level is explained by the factors of each subpopulation and by the factors that tend to increase the level of equality between each and every pair of subpopulations.

Corollary 5.3 *The Gini index of equality includes as a particular case the specification of the Gini index of inequality multi-decomposition.*

Proof: The Gini equality index is:

$$G_e = 1 - G = \frac{\sum_{i=1}^{n}\sum_{r=1}^{n}(x_i + x_r) - \sum_{i=1}^{n}\sum_{r=1}^{n}|x_i - x_r|}{2\mu n^2}. \tag{5.7}$$

Note that the first term, the pure equality, say term A,

$$A = 1 = \frac{\sum_{i=1}^{n}\sum_{r=1}^{n}(x_i + x_r)}{2\mu n^2}, \tag{5.8}$$

is the maximum value for the Gini index of inequality,

$$G = \frac{\sum_{i=1}^{n}\sum_{r=1}^{n}|x_i - x_r|}{2\mu n^2}. \tag{5.9}$$

Following the property of decomposition based on interpersonal comparisons, gathering the income pairs within groups and between groups, term A is subgroup decomposable. Furthermore, we can show that the pure equality is multi-decomposable by both subgroup and income source:

$$\begin{aligned} A =& \frac{\sum_{m=1}^{M}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{r=1}^{n_k}(x_{ik}^m + x_{rk}^m)}{2\mu n^2} \\ &+ \frac{\sum_{m=1}^{M}(2\sum_{k=2}^{K}\sum_{h=1}^{k-1}\sum_{i=1}^{n_k}\sum_{r=1}^{n_h}(x_{ik}^m + x_{rh}^m))}{2\mu n^2} \\ =& \sum_{m=1}^{M} A_w^m + \sum_{m=1}^{M} A_{gb}^m. \end{aligned} \tag{5.10}$$

Following Mussard (2004), we know that the Gini multi-decomposition is given by:

$$\begin{aligned} G =& \underbrace{\sum_{m=1}^{M} \frac{\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{r=1}^{n_k}(x_{ik}^m + x_{rk}^m - 2x_{irk}^{*m})}{2\mu n^2}}_{G_w^m} \\ &+ \underbrace{\sum_{m=1}^{M} \frac{2\sum_{k=2}^{K}\sum_{h=1}^{k-1}\sum_{i=1}^{n_k}\sum_{r=1}^{n_h}(x_{ik}^m + x_{rh}^m - 2x_{irkh}^{*m})}{2\mu n^2}}_{G_{gb}^m}, \end{aligned} \tag{5.11}$$

where G_w^m is the contribution of the m-th source to the within-group Gini index of inequality and where G_{gb}^m is the contribution of the m-th source to the gross between-group Gini index of inequality. Given equations (5.10) and (5.11), we demonstrate that the Gini index of equality multi-decomposition involves the multi-decomposition of the Gini index of inequality as a particular case:

$$G_e = \underbrace{\sum_{m=1}^{M} A_w^m - \sum_{m=1}^{M} G_w^m}_{\sum_{m=1}^{M} G_{ew}^m} + \underbrace{\sum_{m=1}^{M} A_{gb}^m - \sum_{m=1}^{M} G_{gb}^m}_{\sum_{m=1}^{M} G_{egb}^m}. \ \square \tag{5.12}$$

Corollary 5.4 *The distributions of the K subpopulations are K replications, (i.i.d. income distributions) $\Longrightarrow \mu_k = \mu_h = \mu$ ($\forall k, h = 1, \ldots, K$):*

$$G_e = G_{ekk} = G_{ekh} \ (\forall k, h = 1, \ldots, K) \tag{5.13}$$

$$\Longrightarrow G_{ew} = \frac{1}{K} G_e \ ; \ G_{egb} = \frac{K-1}{K} G_e. \tag{5.14}$$

(i) First case: the K subpopulations are K replications, that is, i.i.d. income distributions with different income source distributions, then:

$$G^m_{ekk} = G^m_{ehh} = G^m_{ekh} \ (\forall k, h = 1, \ldots, K); \tag{5.15}$$

$$G_{ew} = \sum_{k=1}^{K} K p_k s_k G^m_{ekk} = \frac{1}{K} G_e; \tag{5.16}$$

$$G_{egb} = \sum_{m=1}^{M} \mathbf{C}^2_K (p_k s_h + p_h s_k) G^m_{ekh} = (1 - \frac{1}{K}) G_e. \tag{5.17}$$

(ii) Second case: the K subpopulations are K replications ($i.d.d.$ income distributions) with M i.i.d. income source distributions, then:

$$G^m_{ekk} = G^m_{ehh} = G^p_{ehh} = G^m_{ekh} \ (\forall k, h = 1, \ldots, K \ ; \ \forall m, p = 1, \ldots, M); \tag{5.18}$$

$$G_{ekk} = M G^m_{ekk} = G_{ekh} = M G^m_{ekh} = G_e; \tag{5.19}$$

$$G_{ew} = \sum_{m=1}^{M} \sum_{k=1}^{K} p_k s_k G^m_{ekk} = MK G^m_{ekk}; \tag{5.20}$$

$$\begin{aligned} G_{egb} &= \sum_{m=1}^{M} \sum_{k=2}^{K} \sum_{h=1}^{k-1} (p_k s_h + p_h s_k) G^m_{ekh} \\ &= \sum_{m=1}^{M} \mathbf{C}^2_k (p_k s_h + p_h s_k) G^m_{ekh} \\ &= M \mathbf{C}^2_k (p_k s_h + p_h s_k) G^m_{ekh}. \end{aligned} \tag{5.21}$$

Proof: It is straightforward. □

Corollary 5.5 *Consider that the M income source distributions are M replications within each subgroup (i.i.d. income source distributions), then:*

$$G_{ew} = \sum_{k=1}^{K} p_k s_k M G^m_{ekk}; \tag{5.22}$$

$$G_{egb} = \sum_{k=2}^{K} \sum_{h=1}^{k-1} (p_k s_h + p_h s_k) M G^m_{ekh}. \tag{5.23}$$

Proof: It is straightforward. □

The multi-decomposition used in this section is only based on the intersection of subgroup and income source decomposition domains. In order to obtain more information about the determinants of equality (and inequality), we present, in the following, a multi-decomposition technique in a multidimensional context.

6 The Multi-Decomposition in a multidimensional context

6.1 The framework

Now, consider we study many dimensions (attributes) such as education level, health level, alphabetization, and other attributes. Suppose we gather the individuals into K subgroups (areas, gender, etc.). Thus, it is possible to apply the Gini index of equality multi-decomposition associated with these dimensions. However, a first problem arises with the comparability of the dimensions and the correlations between them. As Mussard and Xu (2004) showed for the Sen index, we can simply consider a linear econometric modelisation (or a linearisable model) such as individual's income is modelled with M explanatory variables (M attributes), that is, M variables X's describing individual characteristics, social and economic conditions and social policy indicators:

$$x_i = \sum_{m=1}^{M} \alpha_m X_i^m + \varepsilon_i, \tag{6.1}$$

where $X_i^1 = 1$, ε_i is the random error term and i is a subscript for individual i. In model selection, the hypothesis tests, say the Student-t tests, can be implemented to single out the significant variables for explaining the within- and the gross between-group equality.

Theorem 6.1 *The Gini parametric multi-decomposition of the Gini index of equality permits all the couples "within-group/sources" and "between-group/sources" to be computed in a multi-dimensional context, within consistency of the aggregate Gini index of equality exactly.*

Proof: Rewriting $\hat{x}_i^m = \hat{\alpha}_m X_i^m$ ($\forall i = 1, \ldots, n$), we still have a linear structure of income $x_i = \sum_{m=1}^{M+1} \hat{x}_i^m$, where $\hat{x}_i^{M+1} = \varepsilon_i$. Introducing this in the Gini index of equality entails the parametric multi-decomposition of the Gini index of equality:

$$G_e = \sum_{k=1}^{K} \sum_{m=1}^{M} \frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_k} 2\hat{x}_{irk}^{*m}}{2\mu n^2} + 2 \sum_{k=2}^{K} \sum_{h=1}^{k-1} \sum_{m=1}^{M} \frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_h} 2\hat{x}_{irkh}^{*m}}{2\mu n^2}, \tag{6.2}$$

This parametric multi-decomposition yields the contribution of the different dimensions to the within-group and to the gross between-group Gini index of equality. □

Corollary 6.2 *The Gini index of equality involves as a particular case the specification of the parametric Gini index of inequality multi-decomposition.*

Proof: It is similar to the demonstration of corollary (5.3):

$$G_e = \underbrace{\sum_{m=1}^{M} A_w^m - \sum_{m=1}^{M} G_w^m}_{\sum_{m=1}^{M} G_{ew}^m} + \underbrace{\sum_{m=1}^{M} A_{gb}^m - \sum_{m=1}^{M} G_{gb}^m}_{\sum_{m=1}^{M} G_{egb}^m} . \square \tag{6.3}$$

Then, we know if (in)equality is due to a specific attribute within a particular group or if there is some (in)equality between pairs of subgroups for any kind of attribute. For instance, it is possible to state if income equality is due to education, health or alphabetization within the female group or if there exists some income equality between males and females *via* education, health or alphabetization (as in Table 1). We also estimate the weight of the intercept and those of the error term in the within- and the between-group equality.

6.2 Statistical inference

Dealing with linear models confers the opportunity to infer on the couples "within-group/attributes" and "between-group/attributes", i.e., to compute their confidence intervals. Indeed, measuring the variance of these couples, we immediately provide their confidence intervals and their Student-t tests. This is crucial since it produces the significant couples that explain the overall amount of (in)equality. Also, this indicates if the evolution of the (in)equalities are significant over the time. Rewriting the multi-decomposition of the Gini index of equality (6.2) in including the parameters of the regression $\hat{\alpha}_m X_i^m$ yields the relative contribution of the m-th attribute of the k-th group to G_e and the relative contribution of the m-th attribute between the k-th group and the h-th group to G_e:

$$C_{ek}^m = \hat{\alpha}^m \left[\frac{\frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_k} 2X_{irk}^{*m}}{2\mu n^2}}{G_e} \right], \tag{6.4}$$

$$C_{ekh}^m = \hat{\alpha}^m \left[\frac{\frac{2\sum_{i=1}^{n_k} \sum_{r=1}^{n_h} 2X_{irkh}^{*m}}{2\mu n^2}}{G_e} \right]. \tag{6.5}$$

The standard errors of these relative contributions are (see e.g. Morduch and Sicular (2002)):

$$\sigma_{ew}^{m} = \sigma_{\hat{\alpha}^m} \left[\frac{\frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_k} 2X_{irk}^{*m}}{2\mu n^2}}{G_e} \right], \tag{6.6}$$

$$\sigma_{egb}^{m} = \sigma_{\hat{\alpha}^m} \left[\frac{\frac{2\sum_{i=1}^{n_k} \sum_{r=1}^{n_h} 2X_{irkh}^{*m}}{2\mu n^2}}{G_e} \right]. \tag{6.7}$$

Therefore, it is possible to compute all the relative contributions C_{ek}^{m}, C_{ekh}^{m} and to implement the well-known Student-t tests with their confidence intervals [ci] (see Table 2).

Corollary 6.3 *The Gini index of equality does not involve as a particular case the analytical expression of the standard errors of the Gini index of inequality contributions.*

Proof: Given the non-linearity of the variance operator, the computation of σ_{ew}^{m} and σ_{egb}^{m} does not imply those of the Gini index of inequality. But, applying the same procedure provides the standards errors of the relative contribution of the m-th attribute of the k-th group to G and the relative contribution of the m-th attribute between the k-th group and the h-th group to G:

$$\sigma_{w}^{m} = \sigma_{\hat{\alpha}^m} \left[\frac{\frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_k} 2X_{irk}^{*m}}{2\mu n^2}}{G} \right], \tag{6.8}$$

$$\sigma_{gb}^{m} = \sigma_{\hat{\alpha}^m} \left[\frac{\frac{2\sum_{i=1}^{n_k} \sum_{r=1}^{n_h} 2X_{irkh}^{*m}}{2\mu n^2}}{G} \right]. \ \square \tag{6.9}$$

Table 2: Structure of the multidimensional multi-decomposition

Attributes → Groups ↓	Attribute 1	...	Attribute M	Total
(In)equalities in group 1	$C_{(e)1}^{1}$ [ci]	...	$C_{(e)1}^{M}$ [ci]	(.)%
⋮	⋮	⋮	⋮	⋮
(In)equalities in group K	$C_{(e)K}^{1}$ [ci]	...	$C_{(e)K}^{M}$ [ci]	(.)%
(In)equalities between 1 / 2	$C_{(e)12}^{1}$ [ci]	...	$C_{(e)12}^{M}$ [ci]	(.)%
⋮	⋮	⋮	⋮	⋮
(In)equalities between $K-1$ / K	$C_{(e)(K-1)K}^{1}$ [ci]	...	$C_{(e)(K-1)K}^{M}$ [ci]	(.)%
Total	(.)%	...	(.)%	100%

7 Conclusion

When the Gini index of equality is expressed in an extensive form, i.e., including the Gini index of inequality as can be shown in equations (3.16), (3.18), (4.9), (5.12) and (6.3), we obtain a dual decomposable measure, in the sense that the Gini index of equality provides both a decomposition in terms of equality and inequality. Thus, the Gini index of inequality is a particular case of the Gini index of equality in the decomposition context. Therefore, when one intends to decompose the equality coefficient by subgroup, by income source, both by income source and subgroup, and finally both by attribute and subgroup, one also obtains the inequality decompositions. Furthermore, the parametric multi-decompositions yield the computation of all the couples "attributes/within-group" and "attributes/between-group" that explain the overall amount of (in)equality. This is helpful to improve the equalization of opportunities as well as the contemplation of socio-economic policies of redistribution, which can rely on increasing equality and/or reducing inequality.

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