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Abstract

Given the multiplicative decomposition of the Sen index into three commonly used poverty statistics – the poverty rate (poverty incidence), poverty gap ratio (poverty depth) and 1 plus the Gini index of poverty gap ratios of the poor (inequality of poverty) – the index becomes much easier to use and to interpret for economists, policy analysts and decision makers. Based on the recent findings on simultaneous subgroup and source decomposition of the Gini index, we examine possible further decompositions of the Sen index and its components for policy analysis.

Key-words: Gini index, Sen index, Source decomposition, Subgroup decomposition.

JEL Classification: I32, D63, D31.

1 Introduction

Over the last few decades, the literature on inequality and poverty measures has evolved significantly. One of the important developments was the proposal of a poverty measure made by Amartya Sen (1976), now called the Sen index. This index is attractive — easy to understand and convenient for applied research and policy analysis — because of its decomposability into three measures of poverty: incidence (the poverty rate), depth (poverty gap ratio), and inequality (1 plus the Gini index of poverty gap ratios) (see Xu and Osberg (2001)).[1] Naturally, economists and policy analysts would like to know whether it is possible to further decompose the Sen index components according to subgroups (e.g., age group, educational, regional, etc.) or expenditure/income sources (food and non-food expenditure decomposition: income source decomposition, etc.). The advantage for an inequality/poverty measure to possess subgroup and source decomposability is that this allows researchers to measure and, therefore, to appreciate how each of contributing components affects the overall inequality/poverty. This paper will explain how further decomposition of the Sen index can be made with these practical purposes in mind. There are two general approaches to "source/group" decomposition of the Sen index. The first approach is a general decomposing method suitable for any aggregate inequality index (see Chantreuil and Trannoy (1999)) or any aggregate inequality/poverty index (see Shorrocks (1999)) via the mechanism of the Shapley value, which does not lend itself to the unique analytical structure of an index. The second approach, considered in this paper, bases decomposition on the existing analytic structure of an inequality/poverty index. In this paper, we follow the second approach noting that the first approach is readily applicable to the Sen index. That is, we wish to examine the multiplicative decomposition structure of the Sen index and to use the structure for source and subgroup decomposition purposes. In other words, we consider the question of how to implement source/group decomposition to the Sen index and its three components – incidence, depth, and inequality of poverty – jointly. The merit of this approach is that the further decomposition is made on the basis of three distinct but related dimensions of poverty. But this approach is not applicable to any arbitrary inequality/poverty indices as the mechanism of the Shapley value is. The remainder of the paper is organized as follows. In section 2, notation and identification issues are introduced and discussed. Section 3 reviews the principles regarding decomposition, in particular subgroup decomposition, and explains the views of our own on this matter. In section 4, the multidimensional decomposition of the Sen index and its components are analyzed. Finally, section 5 is devoted to the concluding

[1] As shown later in Footnote 2, the Sen index is closed related to another modified Sen index, called the SST index. The discussion here will provide more insight of the Sen index but will not de-emphasize the role of the SST index.

remarks.

2 Notation and identification

Let the number of expenditure/income units, say individuals, in a population be n and the number of the poor individual whose expenditure/income below the poverty line z in expenditure/income be q. In this population there are K distinct subgroups. In subgroup k there are q_k poor individuals among n_k total individuals. The overall poverty rate is $H = \frac{q}{n}$ and the poverty rate for subgroup k is $H_k = \frac{q_k}{n_k}$ with $H = \sum_{k=1}^{K} \frac{n_k}{n} H_k$. Given expenditure/income of the poor individual y_i and the poverty line z, one can define the poverty gap ratio(sometime called relative poverty gap or poverty gap) as

$$x_i = \begin{cases} \frac{z-y_i}{z}, & \forall z > y_i \\ 0, & \end{cases} \tag{1}$$

for all q poor individuals. Then, the vector of poverty gap ratios of the poor is given by: $x_p = [x_1, \ldots, x_i, \ldots, x_q]$. The total expenditure/income of an individual is the sum of all expenditure/income components received by this individual,

$$\sum_{m=1}^{M} y_i^m = y_i. \tag{2}$$

The identification of the poor is based on to whether or not the expenditure/income of an individual y_i falls below the poverty line z. Obviously, this criterion is also applicable to any subgroups. However, when we attempt to analyse the contributions of shortfalls in expenditure/income components to shortfalls in overall income, we have to consider and accept a condition on the poverty line. That is, the poverty line in terms of expenditure/income can be suitably decomposed according to different sources as

$$\sum_{m=1}^{M} z^m = z. \tag{3}$$

How to determine the values of z^m's depending on the norm on how much money/resource an average individual is supposed to receive from source m as minimum levels (z^m's). A simple way to deal with a suitable decomposition of z is to compute the average expenditure source structure of the reference poor population. More specifically,

$$z^m = z\frac{\sum_{i=1}^{q} y_i^m}{\sum_{i=1}^{q} y_i}. \tag{4}$$

Then, the general configuration of the poverty gap ratio involving the source decomposition is

$$x_i^m = \frac{z^m - y_i^m}{z}, \tag{5}$$

where x_i^m is the poverty gap ratio in source m of individual i such that $\sum_{m=1}^{M} x_i^m = x_i$. Such a decomposition of z has an important feature. While x_i is nonnegative, its components x_i^m's can be positive, or zero, or negative implying that the income component y_i^m is less than, or equal to, or greater than the chosen benchmark z^m. Of course, any decomposition of z of slight variations is also permitted. The approach of the similar spirit has been practiced by development economists in the United Nations on the poverty line and its decomposition in terms of so-called balanced food components and their calories. The food poverty line is typically set to around 2200 calories with many of the small components contributing to the total calories. Given the fact that the poverty line can be suitably decomposed, it is possible to define the total income/expenditure shortfall of individual i who is in subgroup k as

$$x_{ik} = \begin{cases} \frac{z - y_{ik}}{z}, & \forall z > y_{ik} \\ 0 & , \end{cases} \tag{6}$$

Now, the poverty gap ratio in income/expenditure source m of individual i in group k is given by

$$x_{ik}^m = \frac{z^m - y_{ik}^m}{z}. \tag{7}$$

Then, the average poverty gap ratio for the population and that for subgroup k are

$$\overline{x}_p = \sum_{i=1}^{q} \frac{x_i}{q} \text{ and } \overline{x}_p^{(k)} = \sum_{i=1}^{q_k} \frac{x_{ik}}{q_k}, \tag{8}$$

respectively, which are related as follows:

$$\overline{x}_p = \sum_{k=1}^{K} \frac{q_k}{q} \overline{x}_p^{(k)}. \tag{9}$$

3 Discussion of subgroup decomposition and source decomposition

First, it is useful to discuss some general principles regarding subgroup decomposition of inequality and poverty measures. It is well known that the Gini index is not subgroup consistent. The concept of subgroup consistency of an inequality measure (SCIM) can be explained as follows. Let p_k be the proportion of population belonging to subgroup k and s_k the income share of subgroup k ($k = 1, \ldots, K$). A measure of inequality I satisfies the subgroup consistency property if

$$I = f(I_1, \ldots, I_k, \ldots, I_K; p_1, \ldots, p_k, \ldots, p_K; s_1, \ldots, s_k, \ldots, s_K), \tag{SCIM}$$

where f is increasing in its first K arguments. In other words, consider a situation where group k has a change in incomes, *ceteris paribus*, such that the mean income and the number of individuals remain constant. The measure of inequality is said to be subgroup consistent if an increase (or a decrease) in group k's inequality leads to an increase (or a decrease) in the overall inequality. In this clearly defined way, the Gini index is not subgroup consistent (see Shorrocks (1980)). Neither is the Gini index component of the Sen index. This indeed posts a challenge. In the similar spirit, there is the concept of subgroup consistency for poverty measures (SCPM) highlighted by Foster and Shorrocks (1991). Let A and B be two sub-distributions within an aggregate distribution formed by these two sub-distributions (A, B), $P(.; z)$ an index of poverty, and $n(A)$ the number of individuals in distribution A. Two distributions A and B may change to distributions A' and B', respectively. Also assume that the number of observations n in each sub-distribution remains unchanged. That is, $n(A) = n(A\text{'})$ and $n(B) = n(B\text{'})$. If poverty increases when distribution A changes to distribution A' so that $P(A\text{'}; z) > P(A; z)$ while poverty in distribution B remains the same, then the overall poverty level increases:

$$P(A\text{'}, B; z) > P(A, B; z). \tag{SCPM}$$

The Gini and Sen indices do not satisfy SCIM and SCPM, respectively. Now the question is: What would be the ground on which subgroup decomposition of the Gini index or the Sen index can be justified? Two points are worth noting in the inequality/poverty measurement literature. First, Pyatt (1976) notes that the Gini index can be interpreted as a measure comparing each and every pairs of individual's incomes. That is quite consistent with the findings in sociology (see Pedersen (2004)), the sense of inequality is very much related to relative positions of individuals in an income distribution. Second, this relative comparison can be extended to the evaluation of poverty. For example, Sen (1973) argues that: "[I]n any pair-wise comparison the man with the lower income can be thought to be suffering from some depression on finding his income to be lower. Let this depression be proportional to the difference in income. The sum total of all such depressions in all possible pair-wise comparisons takes us to the Gini coefficient." The above views highlight an interesting and powerful point. When inequality (or poverty) decreases in one subgroup, say group A, and remains constant in another subgroup, say group B, the overall inequality (poverty) can increase if the individuals in group B feel that their relative rankings are not improving at the same rate from the change in inequality (poverty) of group A. This intuitive argument indeed violates the often-cited subgroup consistency. However, it is accommodated by the Gini index of inequality, and hence the Sen index of poverty intensity, since these two indices employ the Gini social welfare function implicitly (see Xu and Osberg (2001)) and both involve all the interpersonal comparisons (see also Dagum (1998) and the related pair-based measures of income inequality of Kolm (1999)). Because of the above observations, many authors still consider the decomposition of the Gini index, and hence the Sen index, a valuable exercise from theoretical and practical point of view (see Battacharya and Mahalanobis (1967), Rao (1969), Pyatt (1976), Silber (1989), Lerman and Yitzhaki (1991), Lambert and Aronson (1993), Dagum (1997), Deutsch and Silber (1999), for example). It becomes apparent that subgroup consistency is a useful concept but it should not limit our use of other inequality and poverty which do not satisfy SCIM and SCPM but are otherwise attractive in other meaningful ways. The literature shows that the Gini index is decomposable by subgroup although it is not subgroup consistent. The subgroup decomposability of the Gini index leads Xu and Osberg (2001) to conclude that subgroup decomposability of the Sen index[2]

$$\begin{aligned} S &= H\overline{x}_p(1+G) \\ &= \sum_{k=1}^{K} \frac{n_k}{n} H_k \sum_{k=1}^{K} \frac{q_k}{q} \overline{x}_p^{(k)} (1 + (G_w + G_b + G_t)), \end{aligned} \tag{10}$$

where G is the Gini index of poverty gap ratios of the poor, G_w is the contribution of the inequalities within K subgroups, G_b is the contribution of the inequalities among K

[2]In this paper, we will not discuss the SST (S_{SST}) index but this does not diminish the importance of the SST index. The results provided in this paper are still relevant to the SST index as the Sen index and SST index are closely related and have a one-to-one mapping according to Xu and Osberg (2001)

$$S_{SST} = HS + 2H(1-H)\overline{x}_p.$$

That is, given H and $\overline{x}_p$, it is always possible to compute S_{SST} from S and vice versa.

subgroups excluding the overlap between the distributions of these groups, and G_t is the inequalities between K subgroups limited to the overlap between the conditional distributions or the intensity of transvariation (see Gini (1916), Dagum (1959, 1960, 1961, 1997)). An interesting case about the third subgroup component of the Gini index G_t in Eq. (10) is that it is not subgroup consistent but it measures the intensity of transvariation and still makes sense in the subgroup decomposition context — a positive value of this term can be interpreted as a factor contributing to the overall inequality. Of course, this notion is not new at all. It was noted by Gini himself in 1916 and extended by Dagum (1959, 1960, 1961) and many others. As the Gini index is also decomposable by income source, it seems interesting to perform a multiplicative Sen index based on the source Gini decomposition. Shalit and Yitzhaki (1984) and Lerman and Yitzhaki (1985) proposed a particular source Gini decomposition in three components

$$\begin{aligned} G &= \frac{2\mathrm{Cov}(\mathbf{x}, F)}{\overline{x}_p} \\ &= \frac{2\sum_{m=1}^{M}\mathrm{Cov}(\mathbf{x^m}, F)}{\overline{x}_p} \\ &= \sum_{m=1}^{M} \underbrace{\frac{\mathrm{Cov}(\mathbf{x^m}, F)}{\mathrm{Cov}(\mathbf{x^m}, F^m)}}_{R^m} \times \underbrace{\frac{2\mathrm{Cov}(\mathbf{x^m}, F^m)}{\overline{x}_p^m}}_{G^m} \times \underbrace{\frac{\overline{x}_p^m}{\overline{x}_p}}_{\phi^m} \\ &= \sum_{m=1}^{M} C^m, \end{aligned} \tag{11}$$

where $\mathbf{x}$ stands for the vector of poverty gap ratios of the poor, $\mathbf{x^m}$ stands for the vector of poverty gap ratios of the poor related to the mth source, $\overline{x}_p^m$ the average poverty gap ratio of the poor related to the mth source, F the cumulative distribution function (of the poverty gap ratios of the poor), and F^m the cumulative distribution function of the mth source (of the poverty gap ratios of the poor). The first equality is the definition of the Gini index. The second equality of Eq. (11) is based on the fact that $\mathbf{x} = \sum_{m=1}^{M}\mathbf{x^m}$ and $\mathrm{Cov}(\mathbf{x}, F) = \sum_{m=1}^{M}\mathrm{Cov}(\mathbf{x^m}, F)$. The third equality is the result of simple manipulation. The fourth equality represent the definition of $C^m = R^m G^m \phi^m$. On the right-hand side of the third equality of Eq. (11), the first component (R^m) is the relative weight of the Gini mean difference of the mth source; the second term (G^m) is the Gini index for $\mathbf{x^m}$; and the third term (ϕ^m) is the proportion of the mth source in the average poverty gap ratio of the poor. The product of the three terms (C^m) represents the contribution of the mth source to the overall inequality of $\mathbf{x}$. Without bring out R^m, G^m and ϕ^m, it is also possible to construct the Gini index multi-decomposition. It is a simultaneously breakdown of the Gini index by source and subgroup based on Dagum's methodology (1997), which combine the net between-group inequality and the intensity of transvariation as follows

$$G_{gb} = G_b + G_t. \tag{12}$$

The expression gross between-group Gini index (G_{gb}) measures inequality between each and every pair of the overall population in a more complete sense than the standard net between-group Gini index (G_b) which only measures inequality existing among the mean incomes of all subgroups. Then, following Dagum (1997), we have either a 3-term or a 2-term subgroup Gini decomposition as follows

$$G = \sum_{m=1}^{M}(G_w + G_b + G_t) = \sum_{m=1}^{M}(G_w + G_{gb}). \tag{13}$$

The multi-decomposition technique consists to apply a source decomposition to each component of the subgroup decomposition. This entails a 2-term and a 3-term Gini multi-decomposition (see Mussard (2004) and Mussard (2006) respectively), where the income sources represent the causes of the within- and the between-group components

$$G = \sum_{m=1}^{M}(G_w^m + G_b^m + G_t^m) = \sum_{m=1}^{M}(G_w^m + G_{gb}^m), \tag{14}$$

where G_w^m, G_b^m, G_t^m, and G_{gb}^m are respectively the contributions of the mth source to G_w, G_b, G_t and G_{gb}. In the sequel, we only use the 2-term Gini multi-decomposition. The intuition of this approach is presented in the following table.

Table 1: Structure of the 2-term Gini multi-decomposition[3]

Sources → Groups ↓	Source 1	...	Source m	...	Source M	Total
Within-group Inequalities G_w	G_w^1	...	G_w^m	...	G_w^M	G_w
Gross Between-group Inequalities G_{gb}	G_{gb}^1	...	G_{gb}^m	...	G_{gb}^M	G_{gb}
Total	$\mathcal{G}^1$	...	$\mathcal{G}^m$	...	$\mathcal{G}^M$	G

Here $\mathcal{G}^m$ stands for the contribution of the mth source to the overall inequality. Can we use the source/group decomposition of the Gini index of poverty gap ratios and to show that

[3] In the first column, we have G_w^1 which stands for the contribution of the first source to the within-group inequalities, G_{gb}^1 the contribution of the first source to the between-group inequalities, and $\mathcal{G}^m$ the contribution of the first source to the overall inequality (that is different from the contribution C^m in Lerman and Yitzhaki (1985) in (11)). It is also possible to detail these expressions by bringing out the contribution of a source to a particular group or between two precise groups.

the Sen index and its components permit the source/subgroup decomposition? This will be addressed in the next section.

4 The Multi-decomposition of the Sen index and its components

Following Xu and Osberg (2001), rewriting the decomposition in Eq. (10) using Eq. (13) yields

$$S = \sum_{k=1}^{K} \frac{n_k}{n} H_k \ \sum_{k=1}^{K} \frac{q_k}{q} \overline{x}_p^{(k)} \ \left(1 + (G_w + G_{gb})\right). \tag{15}$$

This result is based on the subgroup Gini decomposition.[4] It is possible to extend this result with the income source Gini decomposition and the Gini multi-decomposition.

Proposition 4.1 *If there exists a source partition in both expenditure/income and the poverty line, then the Gini index of poverty gap ratios of the poor is source decomposable or multi-decomposable. It follows that the multiplicative decomposition of the Sen index is permitted.*

Proof: Given that

$$\overline{x}_p = \sum_{k=1}^{K} \frac{q_k}{q} \overline{x}_p^{(k)} \text{ and } \overline{x}_p^{(k)} = \sum_{i=1}^{q_k} \frac{x_i}{q_k} = \sum_{i=1}^{q_k} \frac{\sum_{m=1}^{M} x_i^m}{q_k} \tag{16}$$

$$\iff \overline{x}_p^{(k)} = \sum_{m=1}^{M} \frac{\sum_{i=1}^{q_k} x_i^m}{q_k} = \sum_{m=1}^{M} \overline{x}_p^{m(k)}.$$

The average poverty gap ratio in source m for subgroup k is:

$$\overline{x}_p^{m(k)} = \frac{\sum_{i=1}^{q_k} x_i^m}{q_k}. \tag{17}$$

Then,

$$\overline{x}_p = \sum_{k=1}^{K} \frac{q_k}{q} \overline{x}_p^{(k)} = \sum_{k=1}^{K} \frac{q_k}{q} \sum_{m=1}^{M} \overline{x}_p^{m(k)} = \sum_{k=1}^{K} \sum_{m=1}^{M} \frac{q_k}{q} \overline{x}_p^{m(k)}. \tag{18}$$

Given Eq. (6), (11), (14), (12), (15), and (18), the Sen index can be written as

$$S = \sum_{k=1}^{K} \frac{n_k}{n} H_k \left(\sum_{m=1}^{M} \sum_{k=1}^{K} \frac{q_k}{q} \overline{x}_p^{m(k)} \right) \left(1 + \sum_{m=1}^{M} \left(R^m G^m \phi^m \right) \right) \tag{19}$$

[4] We only use the subgroup Gini decomposition with two elements since the transvariation between groups G_t suffers from a lack of interpretation when we measure inequalities of poverty gap ratio.

or

$$S=\sum_{k=1}^{K}\frac{n_k}{n}H_k\left(\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{q_k}{q}\overline{x}_p^{m(k)}\right)\left(1+\sum_{m=1}^{M}\left(G_w^m+G_{gb}^m\right)\right).\ \square \tag{20}$$

The first result shows that it is possible to obtain a Sen multiplicative decomposition with the source Gini decomposition à la Shalit and Yitzhaki (1984) and Lerman and Yitzhaki (1985). This yields the impact of the mth source and, furthermore, its relative weight (R^m), inequality (G^m), and proportion (ϕ^m) on the overall Sen decomposition. The second result relies on the simultaneous source/group Gini decomposition. This mixture decomposition is potentially useful for applied researchers because it permits source and subgroup decomposition of the determinants of the overall poverty. It is also desirable to maintain the multiplicative structure of the Sen index, but it would be interesting to gauge the contributing shares of incidence, depth and inequality to the aggregate index.[5]

Proposition 4.2 *The change of the Sen index can be linearly decomposed into the changes of incidence, depth and inequality.*

Proof: Given the operator $\Delta\xi := \log\xi_t - \log\xi_{t-1}$, one can show (see Xu and Osberg (2001))

$$\Delta S=\Delta H+\Delta\overline{x}_p+\Delta(1+G). \tag{21}$$

Then,

$$\Delta S=\Delta\sum_{k=1}^{K}\frac{n_k}{n}H_k+\Delta\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{q_k}{q}\overline{x}_p^{m(k)}+\Delta\left(1+\sum_{m=1}^{M}\left(R^mG^m\phi^m\right)\right) \tag{22}$$

or

$$\Delta S=\Delta\sum_{k=1}^{K}\frac{n_k}{n}H_k+\Delta\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{q_k}{q}\overline{x}_p^{m(k)}+\Delta\left(1+\sum_{m=1}^{M}\left(G_w^m+G_{gb}^m\right)\right).\ \square \tag{23}$$

These multi-decompositions of the growth rate of the Sen index are function of the changes in the poverty rate (incidence), in the average poverty gap ratio (depth) and in the Gini index (plus one) of poverty gap ratios (inequality) of the poor.[6] This decomposition is quite appealing from the practical point of view.[7] However, the logarithm do not allow one to capture the different components of the Gini index. We then propose the following approximation.

[5] The following discussion includes concepts and ideas put forward in our previous paper. For the original source, please consult the Chinese version of Mussard and Xu (2004).

[6] The term "one plus the Gini index" can be viewed as an index of inequality belonging to $[1, 2]$, that is, an extended Gini index of inequality.

[7] This result can also be applied to measure the poverty difference between two distributions.

Proposition 4.3 *The change of the Sen index can be approximated by an alternative decomposition into the changes of incidence, depth and inequality in terms of the Gini index of poverty gap ratios of the poor and its components.*

Proof:

$$
\begin{aligned}
\Delta S = {} & \log\left(\sum_{k=1}^{K}\frac{n_k}{n}H_k\right)_t - \log\left(\sum_{k=1}^{K}\frac{n_k}{n}H_k\right)_{t-1} \\
& + \log\left(\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{q_k}{q}\overline{x}_p^{m(k)}\right)_t - \log\left(\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{q_k}{q}\overline{x}_p^{m(k)}\right)_{t-1} \\
& + \log\left(1+\sum_{m=1}^{M}\left(G_w^m+G_{gb}^m\right)\right)_t - \log\left(1+\sum_{m=1}^{M}\left(G_w^m+G_{gb}^m\right)\right)_{t-1}.
\end{aligned} \tag{24}
$$

Given that $\log(1+\xi)\cong\xi$ (the first-order of Taylor's expansion) then

$$
\begin{aligned}
& \log\left(1+\sum_{m=1}^{M}\left(G_w^m+G_{gb}^m\right)\right)_t - \log\left(1+\sum_{m=1}^{M}\left(G_w^m+G_{gb}^m\right)\right)_{t-1} \\
\cong & \left(\sum_{m=1}^{M}\left(G_w^m+G_{gb}^m\right)\right)_t - \left(\sum_{m=1}^{M}\left(G_w^m+G_{gb}^m\right)\right)_{t-1}.
\end{aligned} \tag{25}
$$

Consequently,

$$
\Delta S \cong \Delta\sum_{k=1}^{K}\frac{n_k}{n}H_k+\Delta\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{q_k}{q}\overline{x}_p^{m(k)}+\left(\sum_{m=1}^{M}\left(G_w^m+G_{gb}^m\right)\right)_t-\left(\sum_{m=1}^{M}\left(G_w^m+G_{gb}^m\right)\right)_{t-1}. \tag{26}
$$

Eq. (24) provides an additive approximation of the time variations of the multi-decomposition of the Sen index. In the same way, in order to capture R^m, G^m and ϕ^m, we obtain

$$
\Delta S \cong \Delta\sum_{k=1}^{K}\frac{n_k}{n}H_k+\Delta\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{q_k}{q}\overline{x}_p^{m(k)}+\left(\sum_{m=1}^{M}\left(R^mG^m\phi^m\right)\right)_t-\left(\sum_{m=1}^{M}\left(R^mG^m\phi^m\right)\right)_{t-1}. \tag{27}
$$

It is interesting to note from the Sen multi-decomposition, in particular Eq. (26) and (27), a change in the proportion of the poor and/or the average of expenditure/income shortfalls below the poverty line are positively related to the change of the Sen index. The increase of inequality between periods t and $t-1$ is also positively related to the Sen index's variation. This observation confirms the principle of transfer in this literature on the basis of the changes in the Sen index over time. □

The variations of the Sen index can be captured by the contribution of the inequality of poverty gap ratios in addition to the contributions from the poverty rate and average poverty gap ratio. For instance, an increase (decrease) of the Sen index, *ceteris paribus*, can be caused by an increase (decrease) in the within-group and/or the gross between-group Gini index of poverty gap ratios. Indeed, given that interpersonal comparisons are permitted, each pair of individuals (within groups or between two groups) can compare their source poverty gap ratios. The multi-decomposition of the Sen index enables the policy makers to detect precisely which source components (various items in expenditure/incomes) are mainly responsible for the increasing poverty intensity. Further, the proposed decomposition method allows decision makers to evaluate the impact of an economic policy on the variation of poverty intensity. For example, a change of fiscal policy may lead to the variation of the sources such as tax and transfers across different social groups and ultimately to the variation of poverty intensity. Finally, the multi-decomposition of the Sen index can be used to evaluate the economic policies in a more transparent framework. The way in which the link is established is that we can relate individual poverty gap ratios to J explanatory variables Ω_{ij}'s describing individual characteristics and social and economic conditions that individuals are facing

$$x_i = \sum_{j=1}^{J} \alpha_j \Omega_{ij} + \epsilon_i, \tag{28}$$

where $i = 1, 2, \ldots, n$ represent individual indices, $\Omega_{i1} = 1$, and ϵ_i is the random error term. Here α_j can be estimated by a suitably chosen estimator $\widehat{\alpha}_j$. In model selection, the hypothesis tests can be implemented to single out the significant variables Ω_{ij}'s for explaining the variations in poverty gap ratios x_i's . These variables include, but not limited to, education, work experience, health condition, region, business conditions, social group membership, social policy, social program, and so on. We can also build the similar model for the mth source component as

$$x_i^m = \sum_{j=1}^{J} \alpha_j^m \Omega_{ij} + \epsilon_i^m \tag{29}$$

for $m = 1, 2, \ldots, M$. Given the above parametric specifications of the models, we have the following result.

Proposition 4.4 *The degree of poverty growth attributable to one or more explanatory variables in the aggregate poverty intensity can be measured in the parametric multi-decomposition of the Sen index.*

Proof: By introducing the linear structure of predicted poverty gap ratios $\sum_{j=1}^{J} \widehat{\alpha}_j \Omega_{ij} = \widehat{x}_i$ and $\sum_{j=1}^{J} \widehat{\alpha}_j^m \Omega_{ij} = \widehat{x}_i^m$ into Eq. (26) and (27), we obtain the parametric Sen multi-decompositions

$$\Delta\hat{S} \cong \Delta\sum_{k=1}^{K}\frac{n_k}{n}H_k + \Delta\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{q_k}{q}\hat{x}_p^{m(k)} + \left(\sum_{m=1}^{M}\left(\hat{G}_w^m + \hat{G}_{gb}^m\right)\right)_t - \left(\sum_{m=1}^{M}\left(\hat{G}_w^m + \hat{G}_{gb}^m\right)\right)_{t-1}, \tag{30}$$

$$\Delta\hat{S} \cong \Delta\sum_{k=1}^{K}\frac{n_k}{n}H_k + \Delta\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{q_k}{q}\hat{x}_p^{m(k)} + \left(\sum_{m=1}^{M}\left(\hat{R}^m\hat{G}^m\hat{\phi}^m\right)\right)_t - \left(\sum_{m=1}^{M}\left(\hat{R}^m\hat{G}^m\hat{\phi}^m\right)\right)_{t-1}. \tag{31}$$

This parametric multi-decomposition yields the contribution of the variations in poverty gap ratios and inequality of poverty to the overall variation of the Sen index due to the explanatory variables. □

Corollary 4.5 *The parametric multi-decomposition of the Sen index permits the evaluation of the marginal impact of one or more explanatory variables.*

Proof: If we are interested one or more explanatory variables, we can denote the collection of indices of the explanatory variables by R. For example, if we are interested in the 2nd, 3rd, and 4th explanatory variables, $R = \{2, 3, 4\}$. If we are only interested in the 5th explanatory variable, $R = \{5\}$. Then one can use Eq. (30), we obtain $(\hat{S}_R - \hat{S})/\hat{S}$ where $\hat{S}_R$ is the Sen index with predicated values of the poverty gap ratios component(s) based on R explanatory variable(s) removed but with the predicted values of the poverty gap ratio components based on other explanatory variables kept. By using the relative quantity $(\hat{S}_R - \hat{S})/\hat{S}$, we obtain the proportion of the Sen index value that is attributable to R factor(s). □
The above parametric-decomposition depends on the underlying regression models to link the J explanatory variables to poverty gap ratios. In the similar spirit, we can have the following result which does not depend on the models but on the group membership.

Corollary 4.6 *The subgroup decomposition of the Sen index permits the isolation of the marginal role of each subgroup.*

Proof: If we are interested in the poverty situation of some population groups relative to the whole population, we can denote the collection of groups by R. For example, if we are interested in the 6th, 7th, and 8th groups (or 9th group), $R = \{6, 7, 8\}$ (or $R = \{9\}$). Then, using the expression $(S_R - S/S)$, where S_R is the Sen index computed by dropping the groups indexed by R, we can observe the impact of these groups. □
The results of the parametric multi-decompositions in Prop. 4.3 are based on an approximation, which is good if and only if the Gini index of poverty gap ratios is of a small quantity. In

order to decompose of the growth rate of the Sen index more precisely, we have the following result.

Proposition 4.7 *The growth rate of the Sen index can be approximated by an alternative decomposition into the growth of incidence, depth and inequality.*

Proof: Following Eq. (10), taking the total differential, it is possible to compute the growth rate of the Sen index and its components

$$\frac{dS}{S} = \frac{dH}{H} + \frac{d\overline{x}_p}{\overline{x}_p} + \frac{dG}{1+G}. \tag{32}$$

Given the change of the components between periods $t-1$ and t can be approximated by $\Lambda\xi_t := \xi_t - \xi_{t-1}$, the growth rate of the Sen index between periods $t-1$ and t is

$$\frac{\Lambda S_t}{S_{t-1}} \cong \frac{\Lambda\left(\sum_{k=1}^{K}\frac{n_k}{n}H_k\right)_t}{H_{t-1}} + \frac{\Lambda\left(\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{q_k}{q}\overline{x}_p^{m(k)}\right)_t}{(\overline{x}_p)_{t-1}} + \frac{\Lambda\left(\sum_{m=1}^{M}\left(R^m G^m \phi^m\right)\right)_t}{(1+G)_{t-1}} \tag{33}$$

or

$$\frac{\Lambda S_t}{S_{t-1}} \cong \frac{\Lambda\left(\sum_{k=1}^{K}\frac{n_k}{n}H_k\right)_t}{H_{t-1}} + \frac{\Lambda\left(\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{q_k}{q}\overline{x}_p^{m(k)}\right)_t}{(\overline{x}_p)_{t-1}} + \frac{\Lambda\left(\sum_{m=1}^{M}\left(G_w^m + G_{gb}^m\right)\right)_t}{(1+G)_{t-1}}. \tag{34}$$

This result shows that the growth rate of the Sen index is an increasing function of the growth rate in incidence, the growth rate in depth, and the changes in inequality between $t-1$ and t. More precisely, Eq. (33) and (34) show that the Sen index is an increasing function of the Gini index elements. □

Proposition 4.8 *The elasticity of the Sen index in inequality between periods $t-1$ and t is a function of the elasticities of depth and incidence in inequality between periods $t-1$ and t in addition the changes in inequality itself.*

Proof: Following Eq. (32), the elasticity of the Sen index in inequality can be written as

$$\frac{dS/S}{dG/G} = \underbrace{\frac{dH/H}{dG/G}}_{\zeta_1} + \underbrace{\frac{d\overline{x}_p/\overline{x}_p}{dG/G}}_{\zeta_2} + \frac{dG/(1+G)}{dG/G}, \tag{35}$$

where ζ_1 and ζ_2 stand for the elasticities of incidence and depth in inequality respectively. ζ_1 [ζ_2] measures the percentage change (increase or decline) of the poverty rate (H) [the average poverty gap ratio ($\overline{x}_p$)] with reference to a one percentage change in the Gini index. Rewriting Eq. (35) with the two sorts of decomposition gives

$$\frac{\Lambda S_t/S_{t-1}}{\Lambda G_t/G_{t-1}} \cong \zeta_1 + \zeta_2 + \frac{\left(\sum_{m=1}^{M} R^m G^m \phi^m\right)_{t-1}}{(1+G)_{t-1}} \tag{36}$$

or

$$\frac{\Lambda S_t/S_{t-1}}{\Lambda G_t/G_{t-1}} \cong \zeta_1 + \zeta_2 + \frac{\left(\sum_{m=1}^{M} G_w^m + G_{gb}^m\right)_{t-1}}{(1+G)_{t-1}}. \tag{37}$$

The above expressions suggest that the elasticity of the Sen index in inequality is attributable to the role of the income/expenditure sources (correlation R^m, inequality G^m, and source share ϕ^m) or the interactions source/within-group (G_w^m) and source/between-group (G_{gb}^m). □
Two other elasticities are also available; these are the elasticities of the Sen index in (i) the poverty rate and (ii) the average poverty gap ratios, respectively. The elasticity of the Sen index in inequality is the most important one if one wishes to link poverty with inequality in a redistribution context between $t-1$ and t (see Ravallion (2005)).

Corollary 4.9 *When progressive Pigou-Dalton transfers are implemented between poor individuals without lifting any of them out of poverty, the elasticity of the Sen index in inequality only depends on the Gini index and is inelastic.*

Proof: Suppose that progressive transfers are performed between poor individuals, from the least poor to the most poor such that the poverty rate and average poverty gap ratio remain the same. This implies that $\zeta_1 = 0$ and $\zeta_2 = 0$ in Eq. (36) and (37) resulting

$$\frac{\Lambda S_t/S_{t-1}}{\Lambda G_t/G_{t-1}} = \frac{\left(\sum_{m=1}^{M} G_w^m + G_{gb}^m\right)_{t-1}}{(1+G)_{t-1}} < 1. \tag{38}$$

The elasticity of the Sen index in inequality depends only on $G/(1+G)$ in period $t-1$. Further, the elasticity is less than one or the elasticity is inelastic. □
This result shows that the change in the Sen index is less sensitive to the redistributions in which transfers between poor individuals occurs.

5 Conclusion

In this paper, we show that the multiplicative decomposition of the Sen index of poverty intensity permits the quantitative evaluation of contributions of poverty incidence, depth and inequality as well as the contribution of each group and each income/expenditure source to the overall poverty. This form of multi-decomposition can be extended into a parametric one. Based on this parametric framework statistical inference can be explored to gauge the significance of explanatory variables such as individual characteristics (gender, age, profession, etc.), social and economic conditions (child support benefits, transfers, etc.) and social policy indicators (economic growth, interest rates, etc.). With the similar spirit, we can also examine the marginal roles played by some groups to the overall poverty in the society in our framework. In this paper, we also show the conditions under which the Sen index is insensitive to the redistribution among the poor.

References

Bhattacharya, N. and Mahalanobis B. (1967), "Regional Disparities in Household Consumption in India", *Journal of the American Statistical Association* 62, 143-161.

Chantreuil, F. and Trannoy A. (1999), "Inequality Decomposition Values : The Trade-off Between Marginality and Consistency", DP 9924 THEMA.

Dagum, C. (1959), "Transvariazione Fra Più di Due Distribuzioni", In : Gini, C.(ed.) Memorie di Metodologia Statistica, Vol. II, Libreria Goliardica, Roma. Dagum, C. (1960), "Teoria de la Transvariacion, Sus Aplicaciones a la Economia", *Metron* XX, 1-206.

Dagum, C. (1960), "Teoria de la Transvariacion, Sus Aplicaciones a la Economia", *Metron* XX, 1-206.

Dagum, C. (1961), "Transvariacion en la Hipotesis de Varaibles Aleatorias Normales Multidimensionales", *Proceedings of the International Statistical Institute* 38, Book 4, 473-486, Tokyo.

Dagum, C. (1997), "A New Approach to the Decomposition of the Gini Income Inequality Ratio", *Empirical Economics* 22, 515-531.

Dagum, C. (1998), "Fondements de Bien-être Social et Décomposition des Mesures d'Inégalité dans la Répartition du Revenu", *Economie Appliquée* L1(4), 151-202 [in French].

Deutsch, J. and Silber J. (1999), "Inequality Decomposition by Population Subgroups and the Analysis of Interdistributional Inequality", in Silber, J. (ed.), *Handbook of Income Inequality Measurement*, Kluwer Academic Publishers, 163-186.

Foster, J. E., and Shorrocks A. F. (1991), "Subgroup Consistent Poverty Indices", *Econometrica* 59, 687-709.

Gini, C. (1916), "Il Concetto di Transvariazione e le Sue Prime Applicazioni", in C. Gini, eds., *Giornale degli Economisti e Rivista di Statistica* (1954), 21-44.

Kolm, S-C. (1999), "Rational Foundations of Income Inequality Measurement", in Silber J. (ed.), *Handbook of Income Inequality Measurement*, Kluwer Academic Publishers, p.19-94.

Lambert, P. J. and Aronson R. J. (1993), "Inequality Decomposition Analysis and the Gini Coefficient Revisited", *Economic Journal* 103, 1221-1227.

Lerman, R. and Yitzhaki S. (1985), "Income Inequalities Effects by Income Source: A New Approach and Applications to United States", *Review of Economics and Statistics* 67, 151-156.

Lerman, R. and Yitzhaki S. (1991), "Income Stratification and Income Inequality", *Review of Income and Wealth* 37(3), 313-329.

Mussard, S. (2004), "The Bidimensional Decomposition of the Gini Ratio. A Case Study Italy", *Applied Economics Letters* 11(8), 503-505.

Mussard, S. (2006), "Une Nouvelle Approche de la Décomposition de la Mesure de Gini en Sources de Revenu et la Décomposition en Sous-population: une Réconciliation", *Working Paper* n°05-04, GREDI, Département d'économique, Université de Sherbrooke, *Annales d'Economie et de Statistique*, forthcoming [In French].

Mussard, S., and Xu, K. (2004), "A Note on the Multidimensional Decomposition of Sen's Index", *Chinese Journal of Economic Theory* 1, 201-218.

Pedersen, A. W. (2004), "Inequality as Relative Deprivation - A Sociological Approach to Inequality Measurement", *ACTA Sociologica* 47(1), 31-49.

Pyatt, G. (1976), "On the Interpretation and Disaggregation of Gini Coefficients", *Economic Journal* 86, 243-25.

Rao, V.M. (1969), "Two Decompositions of Concentration Ratio" *Journal of the Royal Statistical Society*, Series A 132, 418-425.

Ravallion, M. (2005), "Inequality Is Bad for the Poor", Work Bank Working Paper WPS3677.

Sen, A. (1973), *On Economic Inequality*, Oxford, Clarendon Press.

Sen, A. (1976), "Poverty: An Ordinal Approach to Measurement", *Econometrica* 44, 219-231.

Shalit, H. and Yitzhaki S. (1984), "Mean-Gini, Portfolio Theory, and the Pricing of Risky Assets", *Journal of Finance* 39, 1449-1468.

Shorrocks, A. F. (1980), "The Class of Additively Decomposable Inequality Measures", *Econometrica* 48, 613-625.

Shorrocks, A. F. (1999) "Decomposition Procedures for Distributional Analysis: A Unified Framework Based on the Shapley Value", *Mimeo*, University of Essex.

Silber, J. (1989), "Factor Components, Population Subgroups and the Computation of the Gini Index of Inequality", *Review of Economics and Statistics* 71, 107-115.

Xu, K. and Osberg, L. (2001), "The Social Welfare Implications, Decomposability, and Geometry of the Sen Family of Poverty Indices", *Canadian Journal of Economics* 35, 138-152.