GRÉDI

Groupe de Recherche en Économie et Développement International

Cahier de recherche / Working Paper
06-09

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Abstract

The aim of this paper is to provide an application of the Shapley Value to decompose financial portfolio risk. Decomposing the sample covariance risk measure yields relative measures, which enable securities of a portfolio to be classified according to risk scales.

Key words: Decomposition, Risk, Shapley, Volatility.
JEL Classification: C3, D31, D63, G11.

1. Introduction

Portfolio risk decomposition gives crucial information for investors concerned with risk measures and decision analysis in portfolio risk management. However, the decomposition into individual risk contributions is not straightforward from the expression of the portfolio variance. The solution proposed by the modern theory of portfolio is the use of the market model to decompose returns into systematic and unsystematic components. Systematic risks can reduce the global risk of a portfolio but cannot indicate the contribution of each security to the global risk of the portfolio without restrictions on hypotheses. Indeed, the estimation of systematic risks using the beta measure from a simple regression model meets some practical problems in estimation. In this paper, we deal with the Shapley value [1953] to decompose the covariance matrix of a portfolio and then to define new variance ratios, which measure the contribution of each security to the overall portfolio risk. The Shapley decomposition method provides new interesting information to classify securities according to different risk scales.

2. Application of the Shapley value to decompose a portfolio

The application of the Shapley value algorithm to decompose inequality measures has been proposed by Trannoy in collaboration with Auvray [1992], with Chantreuil [1999], and with Sastre [2002] and by Shorrocks [1999] for both inequality and poverty measures. The technique allows between- and within-group inequalities to be computed. In financial applications, specific risk measures σ_{w}^2 (say within-security risk) and systematic risk measures σ_{b}^2 (say between-security risk) of a portfolio are derived from the expression of the portfolio variance σ_{p}^2:[1]

$$\sigma_{\mathrm{p}}^2 = \underbrace{\sum_{i=1}^{n} \omega_i^2 \sigma_i^2}_{within-security\ risk} + \underbrace{2\sum_{i=2}^{n}\sum_{j=1}^{i-1} \omega_i \omega_j cov(r_i, r_j)}_{between-security\ risk}, \tag{1}$$

where ω_i is the weight attributed to the i-th security, σ_i^2 the variance of the i-th security and $cov(r_i, r_j)$ the covariance between the returns of the i-th and the j-th shares (r_i and r_j from now on). From (1), it is not possible to measure the risk contributions related to each security to the portfolio variance. Indeed, the decomposition of the between-security risk measure into contribution of each security is not straightforward since multiplicative terms appear in the covariance. We concentrate on the decomposition of between-security risk measures by applying the Shapley value. Let I represent an aggregate risk indicator, and r_k ($k \in K = \{1,2,\dots, n\}$) the security returns of the portfolio, which represent a set of contributory factors which together account for the I value. Then, we can write:

$$I \equiv F(K) \equiv F(r_1, r_2, \dots, r_k, \dots, r_n),$$

where F is a suitable aggregator function of n factors. The decomposition principle consists in assigning contributions C_k to each one of the factors r_k allowing the I value to be expressed as the sum of the factor contributions.

The contribution of each factor may be interpreted as its expected marginal impact when all possible elimination paths are considered. We define $F(S)$ as the value that I may take when the factor r_k have been dropped. In this case, we have $S = K \backslash \{r_k\}$. But Shapley's algorithm considers all possible factor eliminations (one or many factors). Then, the set of returns S ($S \subseteq K$) is the domain of securities remaining after the process of security elimination deriving from Shapley's algorithm and s its cardinality, that is, the number of factors (returns) remaining after the successive eliminations of the different factors. The risk contribution of the k-th factor to a global risk indicator I is given by:

$$C_k(K, F) = \sum_{s=0}^{n-1} \sum_{S \subseteq K \backslash \{r_k\}} \frac{(n-1-s)!\,s!}{n!} \Delta_k F(S), \tag{2}$$

where: $\Delta_k F(S) = F(S \cup \{r_k\}) - F(S)$ with $F(\varnothing) = 0$ (3)

[1] See Mussard and Terraza [2004] for the definition of within- and between-security risk measures of the Gini index.

and $\sum_{k=1}^{n} C_k(K,F) = I.$ (4)

Shorrocks (1999, p. 26) proves that the variance is Shapley decomposable. In this paper, we use this result to measure the contribution of a particular security to the between-security risk, since it is not possible with equation (1). Furthermore, contrary to Shorrocks (1999) we propose a decomposition which is relevant to Markowitz's [1952] theory.

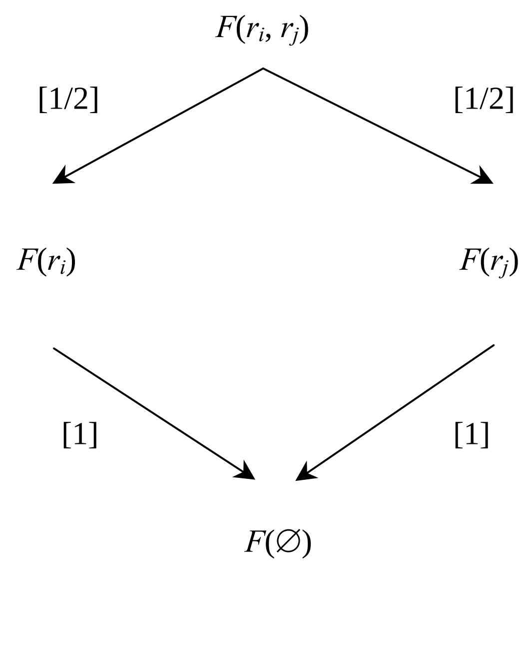
Proposition 1.
The Shapley decomposition of the between-security risk permits to obtain the contribution of each security to the between-security risk index.

Proof.
Let F: $\mathbb{R}^t \times \mathbb{R}^t \rightarrow \mathbb{R}$ be the function representing the weighted covariance between the i-th and the j-th securities, estimated on t observations:[2]

$$F(r_i, r_j) \equiv \omega_j \omega_i cov(r_i, r_j). \quad (5)$$

Given Shapley's algorithm, it is possible to bring out the contribution of each security to the between-security risk. Figure 1 illustrates the Shapley decomposition principle where $F(r_i) \equiv \omega_i^2 cov(r_i, r_i) = \sigma_i^2$, and $F(\varnothing) \equiv 0$ following Shapley's rule.

Figure 1: The Shapley decomposition of the between-security risk
(in brackets, the probabilities)[3]



[2] The Shapley value could have been applied directly to the between-security risk. But in this case, the result is trivial: each security contributes with the same proportion to the between-security risk. This result can be obtained upon the request of the author.
[3] Consider Shapley's elimination principle. Starting from $F(r_i, r_j)$, we have one chance out of two to obtain $F(r_i)$, one chance out of two to have $F(r_j)$, etc.

Following the tree, we obtain the risk contribution of the i-th and the j-th securities to the weighted covariance $F(r_i, r_j)$:

$$C_i = \frac{1}{2}\left(F(r_i, r_j) - F(r_j)\right) + \frac{1}{2}\left(F(r_i) - F(0)\right) \tag{6}$$

$$C_j = \frac{1}{2}\left(F(r_i, r_j) - F(r_i)\right) + \frac{1}{2}\left(F(r_j) - F(0)\right) \tag{7}$$

This entails:

$$C_i + C_j = F(r_i, r_j) = \omega_j \omega_i cov\,(r_i, r_j)\,. \tag{8}$$

Hence, systematic risks σ_{b}^2 can be decomposed applying the Shapley value (2):

$$\sigma_b^2 = 2\sum_{i=2}^{n}\sum_{j=1}^{i-1}\omega_i\omega_j cov\,(r_i, r_j) = 2\sum_{i=2}^{n}\sum_{j=1}^{i-1}(C_i + C_j)\,. \tag{9}$$

Consequently, it is possible to gauge the contribution of each security to the between-security risk index:

$$\sigma_{bi}^2 = 2\sum_{i=2}^{n}\sum_{j=1}^{i-1} C_i\,.\ \blacksquare \tag{10}$$

Proposition 2.
The Shapley decomposition of the between-security risk and the decomposition of the specific risk yield the contribution of the i-th security to the overall amount of the risk portfolio.

Proof.
The within-security risk is naturally decomposable as:

$$\sigma_w^2 = \sum_{i=1}^{n}\omega_i{}^2\,\sigma_i^2\,. \tag{11}$$

Then, the contribution of the i-th security to the specific risk σ_w^2 is:

$$\sigma_{wi}^2 = \omega_i{}^2\,\sigma_i^2\,. \tag{12}$$

Consequently, combining (10) and (12), we obtain the contribution of the i-th security to the portfolio risk σ_{p}^2:

$$\sigma_{\mathrm{p}i}^2 = \sigma_{bi}^2 + \sigma_{wi}^2\,. \tag{13}$$

This Shapley decomposition entails the conception of new risk indicators: the relative contribution of the i-th security related to the specific risk (RVW); the relative contribution of the *i*-th security related to the systematic risk (*RVB*); and the relative contribution of the *i*-th security to the overall amount of the portfolio risk (*RV*):

$$RVW = \frac{\sigma_{wi}^2}{\sigma_p^2}100 \ ; \quad RVB = \frac{\sigma_{bi}^2}{\sigma_p^2}100 \quad ; \quad RV = \frac{\sigma_{\mathrm{p}\,i}^2}{\sigma_{\mathrm{p}}^2}100 \quad . \blacksquare \tag{14}$$

Compared with the beta measure, the *RVB* ratio is less restrictive. Indeed, the beta estimation, using regression method with some restrictive hypotheses, introduces residuals to capture unsystematic risks of securities and then provides very bad approximations of systematic risk measures.

In contrast to this, the Shapley decomposition of the variance is immediate and free-setting distribution. This method produces new interesting ratios for investors. Relative variance measures given by (14) enable to determine the risky securities of a portfolio and to propose a new classification of the securities in accordance with risk scales.

3. Conclusion

Via Shapley's algorithm for *n* securities, we consider the conception of *n*! portfolios. The differences in risk between these portfolios yield the estimation of the marginal contributions of each asset returns to the between-security risk of the portfolio. Then, adding the natural decomposition of the within-security risk measure, we obtain the relative contribution of the securities.

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