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Abstract

For any given order of stochastic dominance, standard concentration curves are decomposed into contribution curves corresponding to within-group inequalities, between-group inequalities, and transvariational inequalities. We prove, for all orders, that contribution curve dominance implies systematically welfare-improving tax reforms and conversely. Accordingly, we point out some undesirable fiscal reforms since a welfare expansion may be costly in terms of particular inequalities.

Keywords and phrases: Concentration curves, Contribution curves, Stochastic dominance, Tax reforms.

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1 Introduction

Yitzhaki and Slemrod (1991) and subsequently Yitzhaki and Thirsk (1990) demonstrated that tax reforms, for pairs of commodities or multiple commodities, can be welfare improving with non-intersecting concentration curves for all additively separable social welfare functions and all increasing $\mathcal{S}$-concave social welfare functions. In 1991, they applied their technique on the extended Gini coefficient. Accordingly, if the concentration curve of good i dominates (lies above) that of good j, in other words, if there are less inequalities in good i than in good j, then an increasing tax on good j combined with a decreasing tax on good i enables decision makers to improve overall welfare or equivalently to decline overall inequalities.

When the population is partitioned in many groups, a usual way to analyze the structure of income inequalities, referring to the Gini index, is to decompose the overall inequality (see e.g. Lerman and Yitzhaki (1991), Dagum (1997a, 1997b) or Aaberge *et al.* (2005) among others) in a within-group index G_W, an average between-group index G_B, and a transvariational index G_T.[1] The latter, being different from a residual, gauges between-group inequalities issued from the groups with lower mean incomes.

In this note, we aim at using the subgroup decomposition technique of the Gini index initiated by Lambert and Aronson (1993) in order to show that standard welfare-improving tax reforms, for pairs of commodities $\{i, j\}$, can be performed with less within-group inequalities, less between-group inequalities in mean, and more transvariational inequalities in good i than in good j. In other words, instead of looking for non-intersecting concentration curves, we provide stronger conditions allowing for welfare-improving tax reforms on goods $\{i, j\}$ by introducing contribution curves for all determinants of overall inequality, namely: within-group, between-group, and transvariational contribution curves. Contrary to the results related to traditional concentration curves (see e.g. Makdissi and Mussard (2006)), we show that, for any order, it is sufficient but not necessary that all contribution curves of good j lie above those of good i, except for the transvariational contribution curve.

The note is attacked as follows. Section 2 reviews Lambert and Aronson's (1993) Gini decomposition. Section 3 introduces notations and definitions. Section 4 explores welfare-improving tax reforms with the concept of contri-

[1] See also the Gini decompositions of Bhattacharya and Mahalanobis (1967), Rao (1969), Pyatt (1976), Silber (1989), Lambert and Aronson (1993), Sastry and Kelkar (1994), Deutsch and Silber (1999).

bution curves for all order of stochastic dominance. Section 5 is devoted to the concluding remarks.

2 Subgroup Decomposition of the Lorenz Curve

In this section, we briefly summarize the results obtained by Lambert and Aronson (1993). Let a population Π of size n and mean income μ be partitioned into K groups: $\Pi_1, \ldots, \Pi_k, \ldots, \Pi_K$ of size n_k and mean income μ_k. The groups are ranked as follows: $\mu_1 \leq \ldots \leq \mu_k \leq \ldots \leq \mu_K$. Assume the individuals are ranked within each Π_k such as the richest person of Π_{k-1} is just positioned before the poorest one of Π_k. Then, the rank of an individual belonging to Π_k is given by: $p_{(p_k)} = \frac{\sum_{i=1}^{k-1} n_i + p_k n_k}{n}$. Therefore, the within-group Lorenz curve $L_W(p_{(p_k)})$ is formalized to compute inequalities within groups:

$$L_W(p_{(p_k)}) = \frac{\sum_{i=1}^{k-1} n_i \mu_i + n_k \mu_k L_k(p_k)}{n\mu}, \tag{1}$$

where $L_k(p_k)$ is the Lorenz curve associated with group Π_k.[2] The Lorenz curve between groups, $L_B(p)$, is obtained by considering that each individual within Π_k earns the mean income of his group μ_k such as the total income $\sum_{k=1}^{K} n_k \mu_k$ is redistributed among the groups:

$$L_B(p_{(p_k)}) = \frac{\sum_{i=1}^{k-1} n_i \mu_i + n_k \mu_k p_k}{n\mu}. \tag{2}$$

The use of these different Lorenz curves yields the overall breakdown of the Gini index (G) in three components: $G = G_W + G_B + G_T$. The contribution of the inequalities within groups (or the within-group Gini) is:

$$G_W = 2 \int_0^1 [L_B(p) - L_W(p)]\, dp. \tag{3}$$

The contribution of the inequalities between groups in mean (or the between-group Gini) is:

$$G_B = 2 \int_0^1 [p - L_B(p)]\, dp. \tag{4}$$

[2] To avoid confusions with further notations, we use $L_W(p_k)$. In the traditional version of Lambert and Aronson's (1993) article, $L_W(\cdot)$ is denoted $C(\cdot)$ with respect to the traditional concentration curve. Indeed, as individuals are ranked by incomes (in ascending order within each group), $C(p)$ measures the proportion of total income received by the first np individuals.

The contribution of the transvariation between groups (or the Gini of transvariation) is:

$$G_T = 2\int_0^1 [L_W(p) - L(p)]\, dp, \tag{5}$$

where $L(p)$ is the Lorenz curve associated with the global population.[3] The transvariation (see Gini (1916), Dagum (1959, 1960, 1961), Deutsch and Silber (1997), among others) brings out the intensity with which the groups are polarized. The greater the transvariation is, or equivalently, the wider the overlap between the distributions is, the lower the polarization may be.

3 Notations and Definitions

The Lorenz curve constitutes the basis of the preceding reasoning of decomposition. As a consequence, for any given consumption good (say j), we gauge the proportion of total consumption of j received by the first np individuals ranked by ascending order of consumption. In the sequel, we use an analogous scheme of decomposition. However, it is related to concentration curves $C^2(p)$, $C_j^2(p)$ being that of good j. We analyze the proportion of total consumption of j received by the first np individuals ranked by ascending order of income. In order to decompose concentration curves, we take recourse to the same lexicographic parade introduced by Lambert and Aronson (1993).

Definition 3.1 *Let p_k be the rank of a person in Π_k according to her income such as $p_{(p_k)} = \frac{\sum_{i=1}^{k-1} n_i + p_k n_k}{n}$, and μ_k^j the k-th group's average consumption of good j such as: $\mu_1^j \leq \ldots \leq \mu_k^j \leq \ldots \leq \mu_K^j$. The between-group concentration curve and the within-group concentration curve of the j-th commodity are expressed as, respectively:*

$$C_{jB}^2(p_{(p_k)}) = \frac{\sum_{i=1}^{k-1} n_i \mu_i^j + n_k \mu_k^j p_k}{n\mu^j} \tag{6}$$

$$C_{jW}^2(p_{(p_k)}) = \frac{\sum_{i=1}^{k-1} n_i \mu_i^j + n_k \mu_k^j C_{jk}^2(p_k)}{n\mu^j}, \tag{7}$$

with $C_{jk}^2(p_k)$ being the concentration curve of group Π_k for good j.

[3]Note that this technique of decomposition is different from those of Dagum (1997a, 1997b), where the inequalities between groups (in mean or transvariation) involve variance and asymmetrical effects between groups, and where G_T is non negative (see also Berrebi and Silber (1987) to learn more about the Gini index with dispersion and asymmetry). Here, $L_W(p) - L(p)$ can be negative, then G_T can also be negative (see also Lerman and Yitzhaki (1991)).

The decomposition technique exhibits different concentration amounts prevailing in a given population. These are related to the number of individuals within each group. Then, one obtains contribution indices, namely, within-group, between-group and transvariational contributions to the overall concentration measure. Indeed, these "population-based measures" explicitly involve the population shares of each Π_k group (see e.g. Rao (1969)). Consequently, these contribution indicators may then be helpful to address issues in the design of indirect tax reforms. For this purpose, we formalize theses contribution indices by initiating the concept of contribution curves. Note that a similar notion, used by Duclos and Makdissi (2005), enables contribution curves of poverty measures to be conceived.[4]

Definition 3.2 *The within-group contribution curve (CC_{jW}), the between-group contribution curve (CC_{jB}), and the transvariational contribution curve (CC_{jT}) of the j-th commodity yield a linear breakdown of the concentration curve of good j:*

$$\begin{aligned}
CC_{jW}(p) &:= C^2_{jB}(p) - C^2_{jW}(p)\\
CC_{jB}(p) &:= p - C^2_{jB}(p)\\
CC_{jT}(p) &:= C^2_{jW}(p) - C^2_{j}(p)\\
\Longrightarrow C^2_j(p) &= p - CC_{jW}(p) - CC_{jB}(p) - CC_{jT}(p).
\end{aligned} \tag{8}$$

The contribution curves coincide with second-degree stochastic dominance.[5] Remark that, integrating any given contribution curve provides a precise contribution to the overall concentration index ($\mathcal{C}$). For instance, $\mathcal{C}_W := 2\int_0^1 [CC_{jW}(p)]\,dp$ yields the absolute contribution of the within-group concentration to the global amount of concentration in good j. In the same manner, one obtains the absolute contribution of between-group and transvariational concentrations, respectively, $\mathcal{C}_B := 2\int_0^1 [CC_{jB}(p)]\,dp$ and $\mathcal{C}_T := 2\int_0^1 [CC_{jT}(p)]\,dp$, such as: $\mathcal{C} = \mathcal{C}_W + \mathcal{C}_B + \mathcal{C}_T$.

For the need of Section 4, s-order concentration curves are introduced.

Definition 3.3 (*Makdissi and Mussard* (2006)). *The first-order concentration curve defined as $C^1_m(p) = x_m(p)/X_m$, is the consumption of good m for an individual at rank p divided by the average consumption of the good. The s-concentration curve is then given by: $C^s_m(p) = \int_0^p C^{s-1}_m(u)\,du$.*

[4] The fact that many persons are affected by poverty or by inequality motivates the use of contribution curve concepts for dominance purposes.

[5] Alternatively, one may consider, as in Aaberge (2004), that first-order dominance is Lorenz dominance. Here, s-order dominance is related to s-concentration curves introduced in Definition 3.3.

4 Fiscal Reform Impacts

Let us define the environment on which we intend to obtain welfare-improving tax reforms. On the one hand, we consider the following rank dependant social welfare function (see Yaari (1987, 1988)):

$$W(F) = \int_0^1 F^{-1}(p)\, v(p)\, dp \tag{H1}$$

where $F^{-1}(p) = \inf\left\{y^E : F\left(y^E\right) \geq p\right\}$ is the left continuous inverse income distribution, y^E the equivalent income, $F\left(y^E\right)$ the distribution of equivalent income, and $v(p) \geq 0$ the frequency distortion function weighting an individual at the p-th percentile of the distribution. On the other hand, we impose this distortion function being continuous and s-time differentiable almost everywhere over $[0,1]$:

$$(-1)^{\ell}\, v^{(\ell)}(p) \geq 0\ ,\ \ \ell \in \{1, 2, \ldots, s\}, \tag{H2}$$

where $v^{(\ell)}(\cdot)$ is the ℓ-th derivative of the $v(\cdot)$ function, $v^{(0)}(\cdot)$ being the function itself. Finally, we restrict our study on the following class of social welfare functions:

$$\widetilde{\Omega}^s := \left\{W(\cdot) \in \{H1 \cap H2\} : (-1)^{\ell} v^{(\ell)}(1) = 0,\ \ \ell \in \{1, 2, \ldots, s\}\right\}. \tag{H3}$$

Suppose the government plans a decreasing tax on good i with an increasing tax on good j, letting his budget constant. This marginal tax reform entails a variation in equivalent income $F^{-1}(p)$ for an individual at rank p:

$$dF^{-1}(p) = \frac{\partial F^{-1}(p)}{\partial t_i} dt_i + \frac{\partial F^{-1}(p)}{\partial t_j} dt_j. \tag{9}$$

As shown by Besley and Kanbur (1988), the change in the equivalent income induced by a marginal change in the tax rate of good i is:

$$\frac{\partial F^{-1}(p)}{\partial t_i} = -x_i(p)\,, \tag{10}$$

where $x_i(p)$ is the Marshallian demand of good i of the individual at rank p in the income distribution. Let M be the number of goods, $m \in \{1, 2, \ldots, M\}$. Suppose a constant average tax revenue, $dR = 0$, where $R = \sum_{m=1}^{M} t_m X_m$ and where X_m is the average consumption of the m-th commodity: $X_m = \int_0^1 x_m(p)\, dp$. Yitzhaki and Slemrod (1991) prove that constant producer prices induce:

$$dt_j = -\alpha \left(\frac{X_i}{X_j}\right) dt_i \text{ where } \alpha = \frac{1 + \frac{1}{X_i}\sum_{m=1}^{M} t_m \frac{\partial X_m}{\partial t_i}}{1 + \frac{1}{X_j}\sum_{m=1}^{M} t_m \frac{\partial X_m}{\partial t_j}}. \tag{11}$$

Wildasin (1984) interprets α as the differential efficiency cost of raising one dollar of public funds by taxing the j-th commodity and using the proceeds to subsidize the i-th commodity. Substituting (11) and (10) in (9) yields:

$$dF^{-1}(p) = -\left[\frac{x_i(p)}{X_i} - \alpha\frac{x_j(p)}{X_j}\right] X_i dt_i. \tag{12}$$

Following Definition 3.1, equation (12) can be rewritten as:

$$dF^{-1}(p) = -\left[C_i^1(p) - \alpha C_j^1(p)\right] X_i dt_i. \tag{13}$$

Consequently, following H1, the variation of social welfare induced by an indirect tax reform is:

$$dW(F) = -X_i dt_i \int_0^1 \left[C_i^1(p) - \alpha C_j^1(p)\right] v(p)\, dp. \tag{14}$$

Theorem 4.1 *A revenue-neutral marginal tax reform, $dt_j = -\alpha\left(\frac{X_i}{X_j}\right) dt_i > 0$ with $\alpha \leq 1$, implies $dW(\cdot) \geq 0$ for all $W(\cdot) \in \widetilde{\Omega}^s$, for any given $s \in \{2, 3, 4, \ldots\}$, if and only if*

$$\begin{aligned} &\alpha CC_{jW}^{s-1}(p) - CC_{iW}^{s-1}(p) \\ &+ \alpha CC_{jB}^{s-1}(p) - CC_{iB}^{s-1}(p) \\ &+ \alpha CC_{jT}^{s-1}(p) - CC_{iT}^{s-1}(p) \geq 0, \text{ for all } p \in [0,1]. \end{aligned} \tag{15}$$

Proof. See the Appendix. ■

Note that the specification of within-group contribution curves brings out the average within-group inequalities. It turns out that, it would be appealing to formalize a taxation technique ensuring decision makers that welfare-improving tax reforms reduce inequalities within all subgroups. Indeed, this condition is not guaranteed in Theorem 4.1, for which within-group inequalities in mean may only be reduced for good j (if αCC_{jW}^{s-1} dominates CC_{iW}^{s-1} for $\alpha \leq 1$). Subsequently, if we were able to construct within-group contribution curves for all groups Π_k, $k \in \{1, 2, \ldots, K\}$, (say $CC_{jW,k}^{s-1}$ for the j-th commodity) and to find a couple of goods $\{i, j\}$ that guarantees dominance between within-group contribution curves for all Π_k, then we could find a welfare-improving tax reform that decreases inequalities within each group. This outcome culminates in the following theorem.

Theorem 4.2 *A revenue-neutral marginal tax reform,* $dt_j = -\alpha \left(\frac{X_i}{X_j}\right) dt_i > 0$ *with* $\alpha \leq 1$*, implies* $dW(\cdot) \geq 0$ *for all* $W(\cdot) \in \widetilde{\Omega}^s$*, for any given* $s \in \{2, 3, 4, \ldots\}$*, if and only if*

$$\begin{aligned} &\sum_{k=1}^{K} \left[\alpha CC_{jW,k}^{s-1}(p) - CC_{iW,k}^{s-1}(p)\right] \\ &+ \alpha CC_{jB}^{s-1}(p) - CC_{iB}^{s-1}(p) \\ &+ \alpha CC_{jT}^{s-1}(p) - CC_{iT}^{s-1}(p) \geq 0, \ \textit{for all } p \in [0,1]. \end{aligned} \tag{16}$$

Proof. See the Appendix. ■

Following Theorem 4.2, a wide range of tax programs are operational with different constraints.

Proposition 4.3 *Three particular solutions of Eq.* (16) *are:*

$$\begin{aligned} \mathcal{S}_1 \ &:= \ \Big\{\alpha CC_{jW,k}^{s-1}(p) \geq CC_{iW,k}^{s-1}(p) \ \forall k \in \{1, 2, \ldots, K\}, s \in \{2, 3, 4, \ldots\} : \\ &\sum_{k=1}^{K} \left[\alpha CC_{jW,k}^{s-1}(p) - CC_{iW,k}^{s-1}(p)\right] \\ &\geq CC_{jB}^{s-1}(p) - CC_{iB}^{s-1}(p) + \alpha CC_{jT}^{s-1}(p) - CC_{iT}^{s-1}(p)\Big\}, \\ \mathcal{S}_2 \ &:= \ \Big\{\alpha CC_{jB}^{s-1}(p) \geq CC_{iB}^{s-1}(p), \ s \in \{2, 3, 4, \ldots\} : \\ &\alpha CC_{jB}^{s-1}(p) - CC_{iB}^{s-1}(p) \\ &\geq \sum_{k=1}^{K} \left[\alpha CC_{jW,k}^{s-1}(p) - CC_{iW,k}^{s-1}(p)\right] + \alpha CC_{jT}^{s-1}(p) - CC_{iT}^{s-1}(p) \geq 0 \Big\}, \\ \mathcal{S}_3 \ &:= \ \Big\{\alpha CC_{jT}^{s-1}(p) \leq CC_{iT}^{s-1}(p), \ s \in \{2, 3, 4, \ldots\} : \\ &\sum_{k=1}^{K} \left[\alpha CC_{jW,k}^{s-1}(p) - CC_{iW,k}^{s-1}(p)\right] + \alpha CC_{jB}^{s-1}(p) - CC_{iB}^{s-1}(p) \\ &\geq CC_{iT}^{s-1}(p) - \alpha CC_{jT}^{s-1}(p)\Big\}. \end{aligned}$$

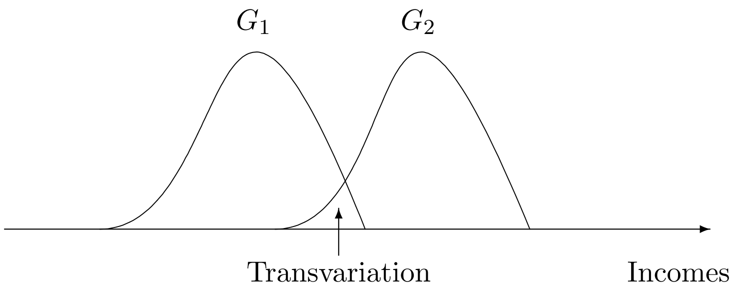
Proof. It is straightforward. ■

($\mathcal{S}_1$) This first solution postulates that all within-group contribution curves of good j dominate those of good i, provided the former is multiplied by α. The condition is that the dominance sum is sufficiently important compared with the remaining terms. Then, an increasing tax on good j, for which the repartition is favorable to rich people, coupled with a decreasing tax on good i produces systematically an overall welfare improvement with alleviation of inequalities within each group, for any s-order stochastic dominance.[6]

($\mathcal{S}_2$) If the between-group contribution curve of the j-th commodity (multiplied by α) lies above that of the i-th commodity, provided Eq. (16) remains positive, then an increasing tax on the j-th commodity coupled with a decreasing tax on the i-th commodity yields necessarily an increase of welfare with a between-group inequality reduction, for any s-order stochastic dominance.

($\mathcal{S}_3$) The third case is an atypical one. Indeed, welfare-improving tax reforms might be performed with a reduction in transvariational inequalities. Nevertheless, as depicted in Figure 1, it is not a desirable issue.

Figure 1. Inequalities of Transvariation



Following Figure 1, when two distributions overlap, inequalities of transvariation are recorded. This particular concept, inspired from Gini (1916) and subsequently developed by Dagum (1959, 1960, 1961), characterizes the income differences between the group of lower mean income (G_1) and that of higher mean income (G_2). Transvariation means that between-group differences in incomes are of opposite sign compared with the difference in the income average of their corresponding group. It is closely connected with economic distances (see e.g. Dagum (1980)), stratification indices (see e.g. Lerman and Yitzhaki (1991)) or polarization measures (see e.g. Duclos, Esteban and Ray (2004)). Therefore, $\mathcal{S}_3$ suggests that welfare-improving tax reforms can be achieved with a growing transvariation (reduction of polarization) between the groups.

[6]Other constraints are available for $\mathcal{S}_1$. For instance, $\alpha CC_{jB}^{s-1}(p) - CC_{iB}^{s-1}(p) + \alpha CC_{jT}^{s-1}(p) - CC_{iT}^{s-1}(p) \geq 0$, may be viewed as a stronger variant. This remark also holds for $\mathcal{S}_2$.

Finally, decision makers can contemplate doing welfare-improving tax reforms subject to the reduction of within-group inequalities, subject to the decline of between-group inequalities or subject to the expansion of transvariational inequalities. However, stronger welfare-increasing tax reforms may be performed in a combinatoric way: αCC_{jW}^{s-1} dominates CC_{iW}^{s-1}, αCC_{jB}^{s-1} dominates CC_{iB}^{s-1}, and CC_{iT}^{s-1} dominates αCC_{jT}^{s-1}.[7] This necessarily implies a welfare gain with alleviation of within-group and between-group inequalities and with transvariational expansion. The reverse being not true.

Application 4.4 *A revenue-neutral marginal tax reform, $dt_j = -\alpha \left(\frac{X_i}{X_j}\right) dt_i > 0$ with $\alpha \leq 1$, that increases Gini social welfare functions under the dominance conditions defined in $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$, enables decision makers to choose between a wide range of inequality aversion parameters ν.*

Proof. The class of functions $W_{SG}(\cdot)$, for which $v(p) = \nu(1-p)^{\nu-1}$, is the well-known family of Gini social welfare functions such as $W_{SG}(\cdot) \in \widetilde{\Omega}^s$. They are concave if $1 < \nu < 2$, convex if $\nu > 2$ and consequently yield exactly the same results as in Theorem 4.2, for any given parameter of inequality aversion. ■

5 Concluding Remarks

The employ of rank dependent social welfare functions is well-suited for the respect of ethical properties such as Pigou-Dalton transfers (Pigou (1912)), a set $\{Ak\}$ of taxation schemes (Gajdos (2002)), uniform α-spreads (Gajdos (2004)), or the principle of positionalist transfers (see e.g. Zoli (1999) and Aaberge (2004)). For the latter, for all $W(\cdot) \in \widetilde{\Omega}^s$, an income transfer from a higher-income individual to a lower-income one (say a progressive transfer) yields a better impact on social welfare as far as individuals' ranks are the lowest as possible. For instance, when $s = 2$, a progressive transfer occurs. For $s = 3$, one gets composite transfers, that is, a progressive transfer arising at the bottom of the distribution combined with a reverse progressive transfer at the top. Higher-order principles can be illustrated with Fishburn and Willig's (1984) general transfer principle, for which composite transfers

[7] The condition $\alpha \leq 1$ yields the set of relevant indirect taxation schemes, see Yitzhaki and Slemrod (1991, p. 483-485). For instance, the case for which $\alpha = 1$ is very useful for applications and implies neither efficiency gain nor efficiency loss for the government, but the indirect taxation program remains relevant, see Makdissi and Wodon (2002, p. 230-231.).

occur both at the bottom and at the top of the distribution. Accordingly, one should analyze, not independently, indirect tax reforms and the implication of the dominance ethical properties resulting from the social welfare function. Therefore, if the s-concentration curve of good i dominates that of good j, then s-order dominance and welfare-improving tax reforms may be interpreted as direct tax programs favorable to lower-income persons coupled with indirect tax programs, such as an increasing tax on the j-th commodity (also favorable to lower-income earners) with a decreasing tax on the i-th commodity, implying an overall welfare expansion.

In a more general fashion, we point out undesirable welfare-improving tax reforms, especially when s-concentration curves are not decomposed. Indeed, as the welfare amplification possesses three inequality counterparts characterized by the contribution curves, it turns out that a fiscal reform may be costly in terms of particular consumption inequalities. Accordingly, it seems reasonable to perform welfare-increasing tax reforms in being aware of the underlying inequality entailments: variation of the inequalities within each group, variation of the inequalities between groups and variation of the transvariational inequalities.

Finally, the methodology allows one to deal with Gini social welfare functions that depend on an inequality aversion parameter. This might contribute to shed more light on the possibility for the social planner to adjust the power of the tax reform in function of the inequality aversion.

Appendix

Proof of Theorem 4.1.
(*Sufficiency*) Integrating successively equation (13) by parts yields:

$$dW(F) = -X_i dt_i (-1)^{s-1} \int_0^1 \left[C_i^s(p) - \alpha C_j^s(p)\right] v^{(s-1)}(p)\, dp. \tag{A1}$$

Given that $-X_i dt_i$ and $(-1)^{s-1} v^{(s-1)}$ are non negative, it is then sufficient to have $C_i^s(p) - \alpha C_j^s(p) \geq 0$, $\forall p \in [0,1]$ with $s \in \{1, 2, \ldots\}$ in order to obtain $dW(\cdot) \geq 0$. Now, we have to decompose the s-order concentration curves C^s into contribution curves CC^l for all $l \in \{1, 2, \ldots s-1\}$, and to use a similar dominance reasoning.
Order $l = 1$:

From equation (A1), an increase of overall welfare is given by $C_i^2(p) - \alpha C_j^2(p) \geq 0$. $C_i^2(p)$ is the traditional concentration curve associated with good i. Indeed, remark that for any given consumption variable x, ranked by ascending order of income, $\int_0^1 x(p)dp$ is an approximation of the arithmetic mean μ. Using the formulae of the sum of trapeze areas, we have: $\int_0^1 x(p)dp = \frac{x_1 p_1}{2} + \frac{(x_1+x_2)(p_2-p_1)}{2} + \ldots + \frac{(1-p_{n-1})(x_{n-1}+x_n)}{2}$. Individual data entail $p_i = \frac{n_i}{n} = \frac{1}{n}$. Then, $\int_0^1 x(p)dp = \frac{x_1.1/n}{2} + \frac{x_1.1/n+x_2.1/n}{2} + \ldots + \frac{1/n.x_{n-1}+1/n.x_n}{2}$ $= \frac{x_1+x_2+\ldots+x_n/2}{n} \cong \mu$. Consequently, the proportion of x detained by the first np individuals is: $P(p) \cong \frac{x_1.1/n+x_2.1/n+\ldots+x_{n-1}.1/n}{\mu} \cong \frac{\sum_i^{n-1} n_i x_i}{n\mu}$, where $p = \frac{n-1}{n}$, and where $P(0) = 0$ and $P(1) = 1$. From Definition 3.1, it is easy to see that $P(p) = C^2(p) \cong 1/\mu \int_0^p x(u)du$. Now remember equation (8): $C^2(p) = p - CC_W(p) - CC_B(p) - CC_T(p)$ and suppose that these contribution curves are first-order curves, that is, $C^2(p) = p - CC_W^1(p) - CC_B^1(p) - CC_T^1(p)$. In order to get $dW \geq 0$ it is sufficient to have $C_i^2(p) \geq \alpha C_j^2(p)$, where $C_i^2(p)$ and $C_j^2(p)$ are respectively concentration curves of goods i and j. Consequently, in order to to have $dW \geq 0$, it is sufficient to match the following condition for all $\alpha \leq 1$:

$$\begin{aligned}
&\alpha CC_{jW}^1(p) - CC_{iW}^1(p) \\
&+ \alpha CC_{jB}^1(p) - CC_{iB}^1(p) \\
&+ \alpha CC_{jT}^1(p) - CC_{iT}^1(p) \geq 0, \text{ for all } p \in [0,1],
\end{aligned} \tag{A2}$$

where CC_{jW}^1 is the first-order within-group contribution curve of good j, CC_{iW}^1 the first-order within-group contribution curve of good i, and so on.

Order $l-1$:

Now assume we have:

$$\begin{aligned}
&\alpha CC_{jW}^{l-1}(p) - CC_{iW}^{l-1}(p) \\
&+ \alpha CC_{jB}^{l-1}(p) - CC_{iB}^{l-1}(p) \\
&+ \alpha CC_{jT}^{l-1}(p) - CC_{iT}^{l-1}(p) \geq 0, \text{ for all } p \in [0,1].
\end{aligned} \tag{A3}$$

Remark that concentration curves of order $l+1$, $C^{l+1}(p)$, are equivalent to integrate $l-1$ times first-order contribution curves:

$$\begin{aligned}
C^{l+1}(p) &= \underbrace{\int_0^p \int_0^u \ldots \int_0^z}_{l-1 \text{ times}} C^2(t)dt \ \ldots \ du \\
&= \underbrace{\int_0^p \int_0^u \ldots \int_0^z}_{l-1 \text{ times}} \left[t - CC_W^1(t) - CC_B^1(t) - CC_T^1(t)\right] dt \ \ldots \ du \\
&= \underbrace{\int_0^p \int_0^u \ldots \int_0^z}_{l-1 \text{ times}} tdt \ldots du - CC_W^{l-1}(p) - CC_B^{l-1}(p) - CC_T^{l-1}(p).
\end{aligned} \tag{A4}$$

<u>Order l:</u>

Integrating (A4) for goods i and j leads to:

$$\begin{aligned}
C_i^{l+2}(p) &= \underbrace{\int_0^p \int_0^u \ldots \int_0^z}_{l \text{ times}} tdt \ldots du - CC_{iW}^{l}(p) - CC_{iB}^{l}(p) - CC_{iT}^{l}(p) \\
C_j^{l+2}(p) &= \underbrace{\int_0^p \int_0^u \ldots \int_0^z}_{l \text{ times}} tdt \ldots du - CC_{jW}^{l}(p) - CC_{jB}^{l}(p) - CC_{jT}^{l}(p).
\end{aligned}$$

Computing the difference between $C_i^{l+2}(p)$ and $C_j^{l+2}(p)$ provided the latter is multiplied by $\alpha \leq 1$, it is then sufficient to match the following condition to obtain $dW \geq 0$:

$$\begin{aligned}
&\alpha CC_{jW}^{l}(p) - CC_{iW}^{l}(p) \\
&+ \alpha CC_{jB}^{l}(p) - CC_{iB}^{l}(p) \\
&+ \alpha CC_{jT}^{l}(p) - CC_{iT}^{l}(p) \geq 0, \text{ for all } p \in [0,1] \,.
\end{aligned} \tag{A5}$$

Equations (A2) and (A5) respect the relationship assumed in (A3). Since (A3) implies (A5), then equation (A5) is true for all $l \in \{1, 2, \ldots, s-1\}$.

(*Necessity*) In order to prove necessity, we consider the set of functions $v\,(p)$, for which the $(s-1)$-th derivative ($v^0(p)$ being $v\,(p)$ itself) is of the following form:

$$v^{(s-1)}\,(p) = \left\{ \begin{array}{cc} (-1)^{s-1}\,\epsilon & p \leq \overline{p} \\ (-1)^{s-1}\,(\overline{p} + \epsilon - p) & \overline{p} < p \leq \overline{p} + \epsilon \\ 0 & p > \overline{p} + \epsilon \end{array} \right. . \tag{A6}$$

Welfare indices whose frequency distortion functions $v(p)$ have the particular above form for $v^{(s-1)}(p)$ belong to $\widetilde{\Omega}^s$. Thus:

$$v^{(s)}(p) = \begin{cases} 0 & p \leq \overline{p} \\ (-1)^s & \overline{p} < p \leq \overline{p} + \epsilon \\ 0 & p > \overline{p} + \epsilon \end{cases} . \tag{A7}$$

Now, imagine equation (A5) with a reverse sign and with $\alpha > 1$:

$$\begin{aligned} &\alpha CC^l_{jW}(p) - CC^l_{iW}(p) \\ &+ \alpha CC^l_{jB}(p) - CC^l_{iB}(p) \\ &+ \alpha CC^l_{jT}(p) - CC^l_{iT}(p) < 0, \ \forall p \in [\overline{p}, \overline{p} + \epsilon], \ \forall \alpha > 1, \end{aligned} \tag{A5'}$$

with ϵ arbitrarily close to 0. For $v(p)$ defined as in (A6), and decomposing (A1) with (A5') for all $\alpha > 1$, we get a tax reform that induces a marginal decrease of welfare: $dW(\cdot) < 0$. Hence, (A5') cannot be for all $p \in [\overline{p}, \overline{p} + \epsilon]$ and $\alpha > 1$. Consequently, $dW(\cdot) \geq 0 \Longrightarrow$ (A5), whenever $\alpha \leq 1$.

Proof of Theorem 4.2.

Remember that the within-group contribution curve $CC_W(p_{(p_k)})$ represents the contribution of the within-group inequalities to the overall inequality. The within-group concentration index $\mathcal{C}_W$ is given by (see e.g. Dagum (1997a, 1997b) for the Gini index case):

$$\mathcal{C}_W = \sum_{k=1}^{K} \frac{n_k \mu_k}{n\mu} . \frac{n_k}{n} \mathcal{C}_k$$

where $\mathcal{C}_k$ is the concentration index of the k-th group:

$$\mathcal{C}_k = \int_0^1 [p_k - C^2_k(p_k)] dp_k.$$

Then, the contribution curve of group Π_k, which represents the contribution of group Π_k to the overall inequality, is:

$$CC_{W,k} = \frac{n_k^2 \mu_k}{n^2 \mu} \left[p_k - C^2_k(p_k) \right],$$

so that:

$$\mathcal{C}_W = \sum_{k=1}^{K} \int_0^1 \frac{n_k^2 \mu_k}{n^2 \mu} \left[p_k - C^2_k(p_k) \right] dp_k.$$

Thus, for the order $l = 1$, the social welfare variation is:

$$\begin{aligned} dW(F) &= X_i dt_i \int_0^1 \left\{ \sum_{k=1}^{K} \left(\alpha CC_{jW,k}^1(p) - CC_{iW,k}^1(p) \right) + p - \alpha p \right. \\ &\quad \left. + \alpha CC_{jB}^1(p) - CC_{iB}^1(p) + \alpha CC_{jT}^1(p) - CC_{iT}^1(p) \right\} v^{(1)}(p)\, dp. \end{aligned}$$

Applying the same induction reasoning as in Theorem 4.1 and the same necessary condition produces the desired result for any given s-order stochastic dominance and for all $\alpha \leq 1$.

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