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Abstract

This paper introduces the concept of Pigou-Dalton transfers between populations of income receivers. Gini's mean difference and Dagum's Gini index between populations are axiomatically derived in order to gauge the impact of within- and between-group Pigou-Dalton transfers on Dagum's measure. We show its sensitiveness for any given transfer in the sense that inequality-reducing transfers are captured when transfers occur from higher-income donors to lower-income recipients, which belong to two distinct populations. Accordingly, we point out the implications of between-group transfers on : the ordering of multivariate majorization, the multivariate stochastic dominance in the sense of multivariate Lorenz ordering and zonotope inclusions, the Gini decomposition, and on the use of the generalized entropy index.

Key-words and phrases : Between-group Gini, Between-group Transfers, Entropy, Pigou-Dalton, Within-group Transfers.

JEL Codes : D31, D63.

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1 Introduction

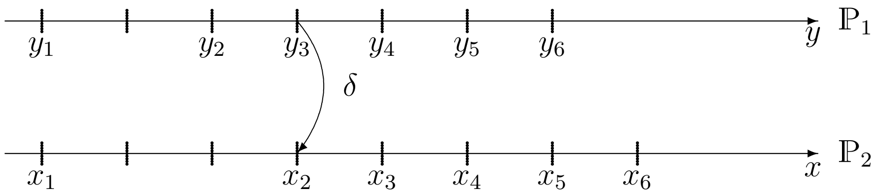
Since Hardy, Littlewood and Pólya (1929, 1934, 1952), mathematician economists have demonstrated the equivalence that prevails between inequality measures, dominance between income distributions and transfer principles. Accordingly, a wide literature as been devoted to such transfers aiming at proposing some well-suited measures of inequality as well as deriving, by means of decision theory, the behavior of decision maximizers under risk and uncertainty (see e.g. Chateauneuf, Gajdos, Wilthien (2002)).

In the spirit of Dalton's transfer (1920), Kolm (1976) proposed the diminishing transfer principle, which postulates that an income transfer, valued to be $\delta > 0$ from a higher-income individual to a lower-income one, yields a better impact on social welfare indices insofar as incomes (x) are the lowest possible, given that individuals' ranking remains unchanged after such transfers. However, as mentioned by Mehran (1976) and Kakwani (1980), these transfers do not involve the difference between individuals' ranks. Hence, they propose the transfer sensitivity property : a transfer of amount $\delta > 0$ occurring at the lower part of the distribution is preferable to a transfer of the same amount at the upper part of the distribution if rank gaps between donors and recipients are equal. The problem is that anything guarantees that income gaps are also equal. In this respect, Chateauneuf *et al.* (2002) suggest the strong diminishing transfer principle : a transfer of amount $\delta > 0$ occurring at the bottom of the distribution is preferable to a transfer of amount $\delta > 0$ at the top of the distribution if rank gaps and income gaps between donors and recipients are equivalent.

Other principles have been proposed, developing some new transfer concepts such as Fleurbaey and Michel's (2001) proportional transfer principle, exhibiting a small loss in the transfer between the donor and the recipient, or Gajdos's (2004) $\alpha-spread$, simulating global changes in a distribution with simultaneous progressive transfers (from one higher-income agent to all lower-income ones) and simultaneous regressive transfers (from the same agent to all higher-income ones) in order to obtain the implications on the decision maker's behavior.

Without exploring new transfer principles, let us now suppose that transfers may occur between individuals that belong to different groups of the population or equivalently between two distinct populations.

Figure 1. Transfer from Population 1 to Population 2



When a progressive transfer is performed from a higher-income individual of Population 1 to a lower-income one of Population 2, many questions arise. How can we measure inequalities between two groups and which inequality-reducing transfers can be characterized ?

In order to tackle these difficulties, we specify particular Pigou-Dalton principles : within-group and between-group transfers. The literature is not totally silent about this topic. On the one hand, inequality measure decompositions, such as entropy or Gini measures (see e.g. Rao (1969), Pyatt (1976), Fei, Ranis and Kuo (1978), Bourguignon (1979), Shorrocks (1980), Cowell (1980), Silber (1989), Lerman and Yitzhaki (1991), Dagum (1997), Wodon (1999), Aaberge, Steinar, Doksum (2005) among others) yield some theoretical issues aiming at bringing out within-group and between-group measures of inequalities. However, these are derived without invoking neither within-group nor between-group transfer axioms. Indeed, transfers are taking into account in order to demonstrate that overall inequality declines with progressive transfers without considering the impacts on within- and between-group inequalities. On the other hand, related topics have been studied since the 80's, that of distance between distributions (initiated by Dagum (1980), Ebert (1984), and Chakravarty and Dutta (1987)), that of polarization (see e.g. Duclos, Esteban and Ray (2004)) and that of deprivation (see e.g. Ebert and Moyes (2000)). Transfers are partly excluded from these researches, hence, some clarifying proposals may be advanced to propose intuitive between-group transfers. This is precisely the aim of this paper.

We first introduce between-group and within-group transfers (Section 2). We axiomatically derive Gini's mean difference (Gini (1912)) and Dagum's Gini index between two populations (Dagum (1987)). Accordingly, we show the implications of between-group transfers on : Dagum's between-group Gini index, the multivariate majorization ordering, and the multivariate Lorenz ordering with zonotope inclusions (Section 3). An application is performed with the Gini decomposition in order to understand the impact of within- and between-group transfers on the variations of the overall Gini index (Section 4). A conclusion follows to highlight the debate between the use of entropy and Gini measures throughout the prism of decomposition techniques (Section 5).

2 Between-group and Within-group Transfers

Notations. Let $\mathbb{D}$ be the set of all income distributions $\mathbb{D} := \bigcup_{n\in\mathbb{N}^*} \mathbb{R}^n$, $\mathbb{D}_+$ being the nonnegative part of $\mathbb{D}$, and $\mathbb{D}_{++}$ its positive part. Let $X := (x_1, \dots, x_i, \dots, x_n)$, $Y := (y_1, \dots, y_i, \dots, y_m) \in \mathbb{D}_+$, be two rank-ordered discrete income distributions : $x_1 \leq x_2 \leq \dots \leq x_n$, $y_1 \leq y_2 \leq \dots \leq y_m$, with $n, m \in \mathbb{N}^*$ and e_i be the n-tuple $(0, \dots, 0, 1, 0, \dots, 0)$ whose only non-zero element occurs at the i-th rank of X ($\widetilde{e}_i$ being the m-tuple of Y) ; $\mathbb{N}^*$ being the set of positive integers. Ω is the set of positive rank-ordered income distributions with equal sizes. $I : \mathbb{D}_+ \longrightarrow \mathbb{R}_+$ is an inequality measure and $I_b : \mathbb{D}_+ \times \mathbb{D}_+ \longrightarrow \mathbb{R}_+$ a between-group inequality measure.

Definition 2.1: ***Pigou-Dalton Transfer.*** *A distribution is obtained from $X \in \mathbb{D}_+$ by a progressive transfer, if a transfer of amount $\delta > 0$ occurs from a higher-income individual x_i to a lower-income one x_j (that is, $x_i > x_j$), letting their position unchanged : $x_{i-1} \leq x_i - \delta$ and $x_j + \delta \leq x_{j+1}$. An index of inequality I satisfies the Pigou-Dalton principle (PD) if :*

$$I\left(X + (e_j - e_i)\delta\right) \leq I(X). \tag{PD}$$

Now, let us define a transfer between members of two distinct populations.

Definition 2.2: ***Between-group Pigou-Dalton Transfer.*** *Let $X, Y \in \mathbb{D}_+$ be two non-identical income distributions, for which a progressive transfer of amount $\delta > 0$ occurs from x_i to y_j, such that $x_i > y_j$, $x_{i-1} \leq x_i - \delta$, $y_j + \delta \leq y_{j+1}$. A between-group inequality index $I_b(X, Y)$ is said to be consistent with the Between-group Pigou-Dalton principle (BPD) if :*

$$I_b\left(X - \delta e_i, Y + \delta \widetilde{e}_j\right) \leq I_b(X, Y). \tag{BPD}$$

This axiom is quite intuitive. If an income donor of X transfers a part of his income to a member of Y, then the income inequality (e.g. the income difference) between the two individuals are lower. On this basis, the BPD axiom postulates that the overall between-group inequalities decline. This will become clearer in Section 5, where it is shown that well-known between-group inequality measures violate BPD.

In a different manner, if a within-group Pigou-Dalton transfer occurs, that is, a transfer between members of a same group, letting their position unchanged within their own group, we require the between-group index is sensitive after such transfers.

Definition 2.3: ***Within-group Pigou-Dalton Transfer.*** *Let $X, Y \in \mathbb{D}_+$ and suppose a progressive Pigou-Dalton transfer of amount $\delta > 0$ occurs within X from x_i to x_j, such that $x_i > x_j$, $x_j + \delta \leq x_{j+1}$, $x_{i-1} \leq x_i - \delta$. A between-group inequality index $I_b(X, Y)$ respects the Within-group Pigou-Dalton principle (WPD) if :*

$$I_b\left(X + (e_j - e_i)\delta, Y\right) \leq I_b(X, Y). \tag{WPD}$$

Remark that many variants of this transfer may be introduced such that the sensitiveness of I_b when a transfer occurs only in Y from y_r to y_h, such that $y_r > y_h$, $y_h + \delta \leq y_{h+1}$, $y_{r-1} \leq y_r - \delta$:

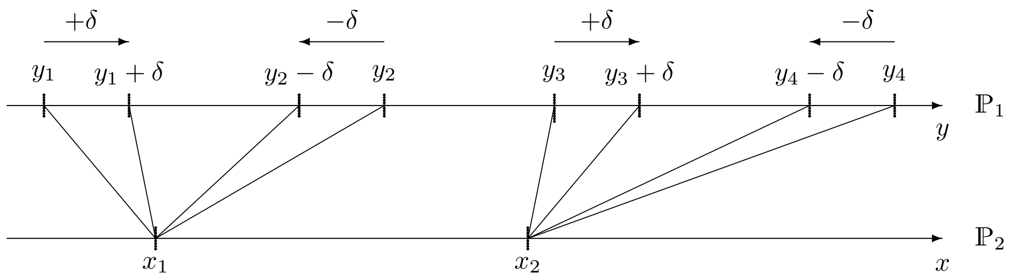
$$I_b\left(X, Y + (\widetilde{e}_h - \widetilde{e}_r)\delta\right) \leq I_b(X, Y). \tag{WPD1}$$

As a corollary of WPD and WPD1, we require that :

$$I_b\left(X + (e_j - e_i)\delta, Y + (\widetilde{e}_h - \widetilde{e}_r)\delta\right) \leq I_b(X, Y). \tag{WPD2}$$

The intuition underlying these transfers is depicted in Figure 2 *infra*. Precisely, after a WPD1 transfer of amount $\delta > 0$ between y_1 and y_2, the poorest individual of $\mathbb{P}_2$ with income x_1 is less distant from $\mathbb{P}_1$ since his income is closer to $y_1 + \delta$ and $y_2 - \delta$ than to y_1 and y_2. In this situation, it makes sense to record a decrease of between-group inequalities. On the contrary, after a transfer of amount $\delta > 0$ between y_3 and y_4, the person with income x_2 may feel indifferent since she is closer to $y_4 - \delta$ but she is more far-off from $y_3 + \delta$. In the light of this, it seems reasonable to pay a particular attention to these between-group inequality measures, which are decreasing after within-group transfers, whereas we will see in Section 5 that many between-group indices are always invariant.

Figure 2. Transfers within Population 1



3 Characterization of Dagum's Between-group Gini index

In order to gauge the impact of within- and between-group transfers on between-group inequalities, we use the traditional axioms of the literature to derive Dagum's Gini index between two groups (see Dagum (1987)), $G : \mathbb{D}_+ \times \mathbb{D}_+ \longrightarrow [0, 1]$. We make this derivation in two times. We first derive the absolute version of the between-group Gini index (homogeneous of degree one), and then, the relative version (homogeneous of degree zero). For this purpose, let us before expose different axioms.

Axiom 3.1: ***Invariance***. *Let $X, Y \in \mathbb{D}_+$, and $\mathbb{1}_x$, $\mathbb{1}_y$ be column vectors of ones with sizes equal to the number of rows in X and Y, respectively. For all $\delta \in \mathbb{R}_{++}$,*

$$I_b(X + \delta\mathbb{1}_x, Y + \delta\mathbb{1}_y) = I_b(X, Y). \tag{INV}$$

Axiom 3.2: ***Linear Homogeneity***. *Let $X, Y \in \mathbb{D}_+$. For all $\lambda \in \mathbb{R}_{++}$,*

$$I_b(\lambda X, \lambda Y) = \lambda I_b(X, Y). \tag{LIN}$$

Axiom 3.3: ***Normalization***. *Let X, $Y \in \mathbb{D}_+$. For all $X = \mathbb{O}_x := (0, 0, \ldots, 0)$ and $Y = \mathbb{1}_y$,*

$$I_b(X, Y) = 1. \tag{NM}$$

Axiom 3.4: ***Symmetry***. *For all* X, $Y \in \mathbb{D}_+$,

$$I_b(X, Y) = I_b(Y, X). \tag{SM}$$

Axiom 3.5: ***Population Principle***. *Let* X, $Y \in \mathbb{D}_+$ *and* $X^{(t)}$, $Y^{(t)}$ *obtained after concatenating* X *and* Y, t *times. For all* $t \in \mathbb{N}^* \setminus \{1\}$,

$$I_b(X^{(t)}, Y^{(t)}) = I(X, Y). \tag{PP}$$

Axiom 3.6: ***Pair-based Decomposition 1***. *Let* $X, Y \in \mathbb{D}_+$. *For all* $x_i \in X$ *and* $y_r \in Y$ *:*

$$I_b(X, Y) = f\left(\sum_{i,j} I_b(x_i, y_j)\right), \tag{DEC1}$$

where i, j *relates all pairwise combinations between* $i \in \{1, 2, \ldots, n\}$ *and* $j \in \{1, 2, \ldots, m\}$, *and* $f(\cdot)$ *is continuous and increasing.*[1]

The concept of pair-based inequality measures (see Kolm (1999)), enables all pairwise income differences to be computed.[2] From these measures, we address the pair-based decomposition axiom, which is different from other decomposition properties we find in the literature of inequality measurement (see Shorrocks (1980)) or mobility measurement (see e.g. Fields and Ok (1996)). This axiom postulates that each person compares her income with each other. This may be viewed as a focus axiom, in which each person feels deprived from higher-income individuals only (see e.g. Bossert and D'Ambrosio (2006)). But in this case, we have a symmetric focus since each person of X compares her situation with higher-income individuals of Y and each person of Y makes the same comparison with members of X. Alternatively, this may be interpreted with a concern for envy : each person of X may envy some individuals of Y and may simultaneously be envied by members of Y (see e.g. Fleurbaey (2006)).

Theorem 3.1 *An absolute between-group inequality index* I_b *satisfies INV, LIN, SM, NM and DEC1 if, and only if, it is the Gini Mean Difference between two groups :*

$$GMD(X, Y) = \frac{\sum_{i=1}^{n} \sum_{r=1}^{m} |x_i - y_r|}{nm}.$$

Proof. See the Appendix. ■

Statistically, the Gini Mean Difference (Gini (1912)) between two populations yields the expected income gap between two individuals drawn at random with replacement, one from X and the other from Y.

[1] Remark that f can be seen as an averaging function or a normalization function. Indeed, in other frameworks, it would be possible to impose some restrictions on f such that symmetry, with $f(x, \ldots, x) = x$, and $f(x_1 + y_1, \ldots, x_n + y_n) = f(x_1, \ldots, x_n) + f(y_1, \ldots, y_n)$, see Aczél (1966, p. 239). Note, in this case, that NM may not be independent from DEC1.

[2] This concept serves to analyze rank-preserving transfer principles and to define intensive and extensive measures of income inequality, see Kolm (1999, p. 53-57).

For the purpose of deriving the relative between-group Gini index, let us introduce the following axioms.

Axiom 3.7: ***Ratio Scale Invariance***. *Let $X, Y \in \mathbb{D}_+$. For all $\lambda \in \mathbb{R}_{++}$,*

$$I_b(\lambda X, \lambda Y) = I_b(X, Y). \tag{RS}$$

As a consequence of NM and RS, we obtain a stronger axiom of normalization.

Axiom 3.8: ***Strong Normalization***. *Let $X, Y \in \mathbb{D}_+$ with $X = \mathbb{O}_x$, and $Y = \mathbb{1}_y$. For all $e \in \mathbb{R}_{++}$,*

$$I_b(X, eY) = 1. \tag{SNM}$$

Axiom 3.9: ***Pair-based Decomposition 2***. *Let $X, Y \in \mathbb{D}_{++}$:*

$$I_b(X, Y) = f\left(\sum_{i,j} C_{I_b}(x_i, y_j)\right), \tag{DEC2}$$

where $f(\cdot)$ is continuous, and where $C_{I_b}(x_i, y_j)$ is the contribution of the pair i, j to the between-group inequality $I_b(X, Y)$.[3]

Theorem 3.2 *A relative between-group inequality index I_b satisfies $SNM \equiv \{RS \cap NM\}$, SM, BPD and DEC2 if, and only if, it is Dagum's between-group Gini index :*

$$G(X, Y) = \frac{\sum_{i=1}^{n} \sum_{r=1}^{n} |x_i - y_r|}{n^2(\overline{x} + \overline{y})}.$$

Proof. See the Appendix. ■

$G : \Omega \times \Omega \longrightarrow [0, 1]$ is Dagum's (1987) Gini index between two populations of income receivers. The strong normalization yields $G(X, Y) \in [0, 1]$, that is, $G(X, Y) = 0$ when the repartition of incomes is egalitarian, and $G(X, Y) = 1$ when it is anti-egalitarian.

Lemma 3.1 *Let $\widetilde{X}, \widetilde{Y}$ be two distributions obtained from two rank-ordered income distributions X, Y by a between-group Pigou-Dalton transfer from $x_i \in X$ to $y_j \in Y$.*
($\imath$) $\forall X, Y \in \mathbb{D}_{++}$, $G(\widetilde{X}, \widetilde{Y}) \leq G(X, Y)$ if, and only if, $r_{x_i/Y} - r_{y_j/X} \geq -1$ and $n \geq m$.
($\imath\imath$) $\forall X, Y \in \Omega$, $G(\widetilde{X}, \widetilde{Y}) < G(X, Y)$.

Then, for groups with equal sizes, $G(X, Y)$ is strictly decreasing when a BPD transfer occurs.

Now, let us turn to particular between Pigou-Dalton transfers, namely, between-quantile transfers (BQT). BQT are performed from $x_i \in X$ to $y_j \in Y$, such that

[3] DEC2 is a weaker requirement of DEC1. The fact that f is continuous enables one to choose between increasing, decreasing or non-linear functions in order to weight differently each income pairwise. This is left for future researches.

$x_i > y_j$, $x_{i-1} \leq x_i - \delta$, $y_j + \delta \leq y_{j+1}$, $\forall X, Y \in \Omega$, for which donors and recipients have the same rank $r_i = r_j$.[4]

Theorem 3.3 *Let $X, Y \in \Omega$, and A be the matrix whose column entries are X and Y, A^T being the transpose of A and A^i the i-th column of A. If there exists a matrix C obtained from A by a finite sequence of between-quantile transfers, then :*
($\imath$) $G(\widetilde{X}, \widetilde{Y}) \leq G(X, Y)$
($\imath\imath$) $C \prec^2 A$, where $\prec^2$ is the (bivariate) majorization ordering
($\imath\imath\imath$) $C^T \preceq^n_L A^T$, where $\preceq^n_L$ is the n-variate Lorenz ordering
($\imath v$) $\sum_{i=1}^2 \phi(C^i) \leq \sum_{i=1}^2 \phi(A^i)$, for all continuous convex functions $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$.

Proof. See the Appendix. ■

This result enables one to derive some results conformably to Hardy, Littlewood and Pólya's spirit, and to draw some connections between stronger between-group Pigou-Dalton transfers and some existing concepts of dominance. On the one hand, between-group quantile transfers imply multivariate stochastic dominance. Indeed, each row of C (representing quantiles) is majorized by those of A. Instead of characterizing this dominance in a one-dimensional context, that is, with the traditional Lorenz curve, a multivariate criterion is employed. Indeed, the majorization ordering can be visualized with the Lorenz *zonotope* of C^T (that is a particular polytope of $\mathbb{R}^{n+1}$), which is included in the Lorenz *zonotope* of A^T.[5] On the other hand, between-group quantile transfers are given by doubly stochastic matrices, which are also $T-$transforms :

$$B = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}, \alpha \in [0,1].$$

Any $T-$transform is given by : $T := \lambda I + (1-\lambda)Q$, $0 \leq \lambda \leq 1$, where I is the identity matrix, and where Q is a permutation matrix that interchanges only two coordinates. C is said to be *chain majorized* by A, that is, $C \preccurlyeq A$, if C is given by : $C = AB$ where B is a product of $T-$transforms. As *chain majorization* $\preccurlyeq$ is equivalent to *majorization* $\prec^d$ if, and only if, $d = 2$ (see Marshall and Olkin (1979), A.2 p. 431), then $G(X, Y)$ can be seen as an order-preserving function. Indeed, as $G(\widetilde{X}, \widetilde{Y}) \leq G(X, Y)$ if $\alpha \in [0, 1]$, then $G(X, Y)$ has a property similar to Schur convexity, but in a bivariate context.

Theorem 3.4 *Let X, Y be $m \times n$ matrices. A differentiable function $\phi : \mathbb{R}^{mn} \longrightarrow \mathbb{R}$ satisfies $\phi(X) \leq \phi(Y)$ for all $X \preccurlyeq Y$ if, and only if,*

[4] Between-quantile transfers may also characterize mobility measures, for which income differences are only computed between the quantiles of each group, see e.g. Fields and Ok (1996).

[5] The equivalence between Pigou-Dalton transfers and the concept of Lorenz dominance between pairs of income distributions F and G is well established in the sense that there is less inequality in distribution F if its corresponding Lorenz curve lies nowhere below that of G. This dominance criterion is here extended in the dimension $n+1$, where the Lorenz *zonotope* is the natural extension of the Lorenz curve (see Koshevoy and Mosler (1996)).

($\imath$) $\phi(X) \leq \phi(X\Pi)$ *for all* $n \times n$ *permutation matrices* Π,
($\imath\imath$) $\sum_{i=1}^{m}(x_{ik} - x_{ij})\left[\frac{\partial\phi(X)}{\partial x_{ik}} - \frac{\partial\phi(X)}{\partial x_{ij}}\right] \geq 0$.

Proof. Rinott (1973). ■

Furthermore, as $G(X, Y)$ respects the majorization ordering, the index possesses the following property.

Theorem 3.5 *Let* X, Y *be* $m \times n$ *matrices. If* $X \prec^n Y$, *then* $\phi(X) \leq \phi(Y)$ *for all* $\phi : \mathbb{R}^{mn} \longrightarrow \mathbb{R}$ *which are symmetric and convex in the sense that :*
($\imath$) $\phi(X) \leq \phi(X\Pi)$ *for all* $n \times n$ *permutation matrices* Π,
($\imath\imath$) $\phi(\alpha X + (1-\alpha)Y) \leq \alpha\phi(X) + (1-\alpha)\phi(Y)$, $0 \leq \alpha \leq 1$.

Proof. Marshall and Olkin (1979, C.3 p. 435). ■

In the case of within-group transfers, the symmetric result of Theorem (3.3) can be obtained. Let us before expose an intermediate result.

Lemma 3.2 *Let* $\widetilde{X}, \widetilde{Y}$ *be two distributions issued from two rank-ordered income distributions* $X, Y \in \mathbb{D}_+$ *by a within-group Pigou-Dalton transfer from* $x_i \in X$ *to* $x_j \in X$ *and from* $y_i \in Y$ *to* $y_j \in Y$, *thus :* $G(\widetilde{X}, \widetilde{Y}) \leq G(X, Y)$.

Proof. See the Appendix. ■

Lemma (3.2) shows that $G(X, Y)$ respects the within-group Pigou-Dalton transfers WPD2. The proofs of WPD and WPD1 are left for the reader.

Theorem 3.6 *Let* $X, Y \in \Omega$ *and* A *the matrix whose column entries are* X *and* Y, A^T *being the transpose of* A *and* A^{Ti} *the i-th column of* A^T. *If there exists a matrix* C *obtained from* A *by a finite sequence of within-group Pigou-Dalton transfers in* X *and by a finite sequence of within-group Pigou-Dalton transfers in* Y, *then :*
($\imath$) $G(\widetilde{X}, \widetilde{Y}) \leq G(X, Y)$
($\imath\imath$) $C^T \prec^n A^T$, *where* $\prec^n$ *is the (n-variate) majorization ordering*
($\imath\imath\imath$) $C \preceq_L^2 A$, *where* $\preceq_L^2$ *is the bivariate Lorenz ordering*
($\imath v$) $\sum_{i=1}^{n}\phi(C^{Ti}) \leq \sum_{i=1}^{n}\phi(A^{Ti})$, *for all continuous convex functions* $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Proof. Analogous to Theorem (3.3). ■

The sequence of within-group Pigou-Dalton transfers is characterized by the product $C^T = A^T B$, where B is an $n \times n$ doubly stochastic matrix. The sum over any given line or column of each element of B, $b_{ij} \in [0, 1]$ is valued to be 1. In this case, $G(\widetilde{X}, \widetilde{Y}) \leq G(X, Y)$. On the contrary, when B is a doubly stochastic matrix but not a permutation one, that is $b_{ij} \in (0, 1)$, the respect of WPD2 is strict. In each case, the majorization ordering $C^T \prec^n A^T$ is respected. This dominance criterion is characterized by the Lorenz *zonotope* of C which is included in that of A, the inclusion being not strict whenever $b_{ij} \in [0, 1]$.[6]

[6] See Horn and Johnson (1991, Theorem (4.3.33)) for the equivalence between the majorization ordering and the existence of doubly stochastic matrices.

4 Application to the Gini Decomposition

As shown in the previous sections, the between-group Gini index is sensitive to between-group Pigou-Dalton transfers. Then, a natural application to the Gini decomposition is of some interest, since the between-group Gini index constitutes an element of the well-known Gini coefficient $\mathcal{G}$.

The Gini decomposition has been extensively studied (see e.g. Battacharya and Mahalanobis (1967), Rao (1969), Pyatt (1976), Silber (1989), Lerman and Yitzhaki (1991), Lambert and Aronson (1993), Sastry and Kelkar (1994), Dagum (1997), Wodon (1999), Aaberge, Steinar, Doksum (2005) among others) but few of these researches yield a full characterization of the Gini index with respect to the nature of the transfers. Transfers are usually invoked in order to ensure Schur convexity to be fulfilled for overall indices, that is, overall inequality declines with progressive transfers. Nevertheless, this is made without apprehending neither the impact of within-group transfers nor that of between-group transfers on global inequality, whereas the overall index is a weighted average of within- and between-group indices.

Let x_{ik} be the income of the i-th individual $i \in \{1, 2, \ldots, n\}$ of the group Π_k with $k, j \in \{1, 2, \ldots, K\}$. Let μ and μ_k be respectively the income average of the overall population (of size n) and that of group Π_k (of size n_k). The overall Gini index is :

$$\mathcal{G} = \frac{\sum_{i=1}^{n} \sum_{r=1}^{n} |x_i - x_r|}{2n^2\mu}.$$

The within-group Gini index of group Π_k and the Gini index between the groups Π_k and Π_j are, respectively :

$$\mathcal{G}_{kk} = \frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_k} |x_{ik} - x_{rk}|}{2n_k^2\mu_k}, \ \mathcal{G}_{kj} = \frac{\sum_{i=1}^{n_k} \sum_{r=1}^{n_j} |x_{ik} - x_{rj}|}{n_k n_j(\mu_k + \mu_j)}.$$

Theorem 4.1 *The Gini index is a weighted average of within- and between-group Gini indices :*

$$\mathcal{G} = \underbrace{\sum_{k=1}^{K} p_k s_k \mathcal{G}_{kk}}_{\mathcal{G}_w} + \underbrace{\sum_{k=2}^{K} \sum_{j=1}^{k-1} (p_k s_j + p_j s_k) \mathcal{G}_{kj}}_{\mathcal{G}_{gb}}, \quad (1)$$

where p_k and s_k are respectively population share of group k ($\frac{n_k}{n}$) and income share of group k ($\frac{n_k \mu_k}{n\mu}$).

Proof. See Dagum (1997). ■

Remark that this two-term decomposition follows directly from the pair-based decomposition axiom (DEC2). Indeed, adapting DEC2 for the overall Gini index yields : $\mathcal{G}(X, X) = f\left(\sum_{i,j} C_{\mathcal{G}}(x_i, x_j)\right)$, where $C_{\mathcal{G}}(x_i, x_j) = \frac{|x_i - x_j|}{2n^2\mu}$, and where $f(x) = x$. Bringing together the income pairs within each group ($\mathcal{G}_w$) and the income pairs between each group pairwise ($\mathcal{G}_{gb}$) yields a within-group Gini index and a gross between-group

Gini index. It is well-known that the Gini index is decomposable in three components : the within-group element ($\mathcal{G}_w$) plus the (gross) between-group element ($\mathcal{G}_{gb}$), which is in turn decomposed into between-group inequalities ($\mathcal{G}_{nb}$) measuring inequalities between the mean incomes of the distribution of each group, and a transvariational element ($\mathcal{G}_t$) that gauges inequalities of overlap between these distributions. For simplicity and without loss of generality, we focus on the two-term Gini decomposition (1).[7] This approach avoids to impose a crude structure to the between-group index and nonsensical responses to within- and between-group transfers (see Section 5).

Theorem 4.2 *Let $\widetilde{\mathcal{G}}$ and $\mathcal{G}$ be respectively ex-post and ex-ante Gini coefficients computed on the overall population. The same distinction applies for $\mathcal{G}_w$, $\widetilde{\mathcal{G}}_w$ and $\mathcal{G}_{gb}$, $\widetilde{\mathcal{G}}_{gb}$. If $\widetilde{X}_k, \widetilde{X}_j \in \mathbb{D}_+$ are obtained from X_k, X_j by a between-group Pigou-Dalton transfer from $x_{ik} \in X_k$ of group Π_k to $x_{rj} \in X_j$ of group Π_j, then :*
($\imath$) $\widetilde{\mathcal{G}} < \mathcal{G}$, $\widetilde{\mathcal{G}}_w < \mathcal{G}_w$, $\widetilde{\mathcal{G}}_{gb} < \mathcal{G}_{gb}$ if $r_{x_{ik}} > r_{x_{rj}}$ and $r_{x_{ik}/X_j} \geq r_{x_{rj}/X_k} - 1$,
($\imath\imath$) $\widetilde{\mathcal{G}} < \mathcal{G}$, $\widetilde{\mathcal{G}}_w > \mathcal{G}_w$, $\widetilde{\mathcal{G}}_{gb} < \mathcal{G}_{gb}$ if $r_{x_{ik}} < r_{x_{rj}}$ and $r_{x_{ik}/X_j} \geq r_{x_{rj}/X_k} - 1$.

Proof. See the Appendix. ■

The fact that a progressive transfer diminishes the overall Gini index is a well-known result. Nevertheless, it is not clear how may vary inequalities within groups and between groups in accordance with the nature of the transfers. In Theorem (4.2), different scenarios of between-group transfers are itemized. ($\imath$) There is simultaneously less within-group and less between-group inequalities. ($\imath\imath$) There is more within-group inequalities, which are compensated for lower between-group inequalities.[8]

Theorem 4.3 *Let $\widetilde{X}_k, \widetilde{X}_j \in \mathbb{D}_+$ being obtained from X_k, X_j by a within-group Pigou-Dalton transfer from $x_{ik} \in X_k$ to $x_{rk} \in X_k$ and from $x_{ij} \in X_j$ to $x_{rj} \in X_j$. Then :*
($\imath$) $\widetilde{\mathcal{G}} < \mathcal{G}$, $\widetilde{\mathcal{G}}_w < \mathcal{G}_w$, $\widetilde{\mathcal{G}}_{gb} < \mathcal{G}_{gb}$ if $\exists \ell \in \{1, 2, \ldots, n_k\}$, $\exists t \in \{1, 2, \ldots, K\} : x_{ik} < x_{\ell t} < x_{rk}$ and/or $\exists z \in \{1, 2, \ldots, n_k\}$, $\exists h \in \{1, 2, \ldots, K\} : x_{ij} < x_{zh} < x_{rj}$.
($\imath\imath$) $\widetilde{\mathcal{G}} < \mathcal{G}$, $\widetilde{\mathcal{G}}_w < \mathcal{G}_w$, $\widetilde{\mathcal{G}}_{gb} = \mathcal{G}_{gb}$ if $\nexists \ell \in \{1, 2, \ldots, n_k\}$, $\nexists t \in \{1, 2, \ldots, K\} : x_{ik} < x_{\ell t} < x_{rk}$ and $\nexists z \in \{1, 2, \ldots, n_k\}$, $\nexists h \in \{1, 2, \ldots, K\} : x_{ij} < x_{zh} < x_{rj}$.

Proof. See the Appendix. ■

From Lemma (3.2), the Gini index between two groups $\mathcal{G}_{kj}$ diminishes if a foreign person lies between the donor and the recipient. Consequently, the overall inequality declines : ($\imath$) with less within-group and less between-group inequalities, or ($\imath\imath$) with less within-group inequalities only.[9]

[7] Rao's (1969) Gini decomposition was the first attempt defining a between-group inequality measure differently from a simple index that measures the differences between mean incomes only. Dagum (1997) extends this idea and proves notably that $G(X, Y)$ captures variance and asymmetrical effects between groups, whereas other measures fail.

[8] Remember that the condition imposed on the conditional ranks can be dropped if $X_j, X_k \in \Omega$, see Lemma (3.1).

[9] Theorem (4.3) does not deal with WPD1 and WPD2. Obviously, we obtain the same results.

5 Concluding Comments

The generalized entropy index I_β (see e.g. Shorrocks (1980), Cowell (1980)) is a well-known inequality measure, which is intensively used for empirical investigations :

$$I_\beta = \frac{1}{\beta(\beta+1)n} \sum_{i=1}^{n} \frac{y_i}{\mu} \left[\left(\frac{y_i}{\mu} \right)^\beta - 1 \right], \ \beta \in \mathbb{R}. \tag{2}$$

The generalized entropy is decomposed such as : $I_\beta = I_{\beta w} + I_{\beta b}$, where $I_{\beta w}$ and $I_{\beta b}$ stand respectively for within- and between-group entropy measures, the latter being a function of the number of persons in each group n_k, $k \in \{1, \ldots, K\}$ and on the income average of each group μ_k. Let $I_{\beta wk}$ represent the inequality in group k. We then obtain :

$$I_{\beta w} = \sum_{k=1}^{K} \frac{n_k \mu_k}{n\mu} \left(\frac{\mu_k}{\mu} \right)^\beta I_{\beta wk}, \ \beta \in \mathbb{R}, \tag{3}$$

$$I_{\beta b} = \frac{1}{\beta(\beta+1)} \sum_{k=1}^{K} \frac{n_k \mu_k}{n\mu} \left[\left(\frac{\mu_k}{\mu} \right)^\beta - 1 \right] = I_\beta \Bigg(\underbrace{\mu_1, \ldots, \mu_1}_{n_1 \text{ times}}, \ldots, \underbrace{\mu_K, \ldots, \mu_K}_{n_K \text{ times}} \Bigg). \tag{4}$$

Equation (4) is derived from the additive decomposability axiom (see Shorrocks (1980)) postulating that each individual earns the mean income of his corresponding group. The generalized entropy comprises the well-known Theil index ($\beta \to 0$), Hirschman-Herfindahl index ($\beta \to 1$), and Bourguignon index ($\beta \to -1$). In the points described *infra*, we expose some limitations underlying the use of between-group measures equivalent to $I_{\beta b}$.

($\imath$) $I_{\beta b}$ is invariant to within-group Pigou-Dalton transfers (WPD, WPD1, WPD2) since it preserves the mean income of the group in which transfers occur (see Table 1). Contrary to this, $G(X_h, X_k)$ is decreasing (not strictly), because the income repartition resulting from WPD, e.g. in group Π_h, may produce some perturbations in group Π_k. This perturbation may incorporate a concern for more exclusion, more deprivation, or more envy expressed from the members of Π_k.

($\imath\imath$) Imagine a progressive transfer of amount $\delta > 0$ from a higher-income individual of group Π_k to a lower-income one of Π_h, coupled with a progressive transfer of the same amount from a higher-income person of Π_h to a lower-income one of Π_k. Obviously, this type of transfer does not affect the mean income of each group, namely, the principle of between-group Pigou-Dalton transfers which are mean-preserving (BPDMP, see Table 1). If the mean incomes remain unchanged, there is no reason to record no variation of between-group inequalities. Even so, $I_{\beta b}$ is always silent about finite sequences of BPDMP. On the contrary, $G(X_h, X_k)$ is decreasing.

($\imath\imath\imath$) When a progressive between-group Pigou-Dalton transfer is itemized between groups Π_h and Π_k, in a given quantile (BQT), $I_{\beta b}$ increases or decreases, whereas $G(X_h, X_k)$ is systematically decreasing.

(iv) Suppose $\mu_h < \mu_k$ and $n_h = n_k$ and assume that a progressive between-group Pigou-Dalton transfer occurs from x_{ih} of Π_h to x_{rk} of Π_k. As $x_{ih} > x_{rk}$ but $\mu_h < \mu_k$, the transfer increases $I_{\beta b}$. This violates the BPD axiom and may violate our basic intuition of between-group inequality measurement. The condition one has to impose to get a reducing $I_{\beta b}$ is that the donor must belong to the group with higher mean income. This does make sense in some circumstances, but it is a too demanding requirement (included in BPD). In this particular case, the overall index I_β decreases, but it is always a one-way decline : $\widetilde{I}_{\beta w} < I_{\beta w}$ and $\widetilde{I}_{\beta b} > I_{\beta b}$ with $I_{\beta w} - \widetilde{I}_{\beta w} > \widetilde{I}_{\beta b} - I_{\beta b}$. On the contrary, $G(X_h, X_k)$ is systematically decreasing, implying $\widetilde{\mathcal{G}}_{gb} < \mathcal{G}_{gb}$. This may occur either with $\widetilde{\mathcal{G}}_w < \mathcal{G}_w$ or with $\widetilde{\mathcal{G}}_w > \mathcal{G}_w$, since a between-group transfer changes the income repartition in the donor's group as well as in the recipient's one.

Table 1. Within-group and Between-group Pigou-Dalton transfers[**]

Transfers → Indices ↓	WPD	WPD1	WPD2	BPD	BPDMP	BQT
$G(X_h, X_k)$	$\leq$	$\leq$	$\leq$	$<$	$<$	$<$
$\mathcal{G}_{gb}$	$\leq$	$\leq$	$\leq$	$<$	$<$	$<$
$\mathcal{G}_w$	$<$	$<$	$<$	$<$ or $>$	$<$ or $>$	$<$ or $>$
$I_{\beta b}$	$=$	$=$	$=$	$<$ or $>$	$=$	$<$ or $>$
$I_{\beta w}$	$<$	$<$	$<$	$<$ or $>$	$<$	$<$ or $>$

[*] For simplicity, assume groups with equal sizes : $X_h, X_k \in \Omega$.
[**] $\leq$: decreasing or equal ($<$ decreasing) after transfer.
[**] $\geq$: increasing or equal ($>$ increasing) after transfer.
[**] $=$: invariant after transfer.

Actually, the additive decomposability yields $I_{\beta b}$ for which the income distributions of each group are obtained with finite sequences of within-group transfers in each group (WPD2). This allows to achieve egalitarian (but not necessarily identical) income distributions. Then, using $I_{\beta b}$ to address empirical applications can be seen as a particular exercise.[10] Furthermore, as $I_{\beta b}$ is not sensitive to WPD transfers (see the line $I_{\beta b}$ in Table 1), if we assume by duality that the welfare counterpart of $I_{\beta b}$ is also invariant, then we do not match all intuitive results developed by Chateauneuf *et al.* (2002) about decision making under risk and uncertainty.

Finally, the use of pair-based decomposability (DEC1/DEC2) confers the ability to perform inequality-reducing tax reforms by taxing one group and using the proceeds to subsidize another group. Then, in the spirit of Hardy, Littlewood and Pólya's works (1929, 1934, 1952), the pair-based decomposability allows one to address a close interrelation between : between/within-group transfers, the majorization ordering, the multivariate Lorenz ordering, and Schur convexity in the sense of Rinott's (1973) theorem.

[10]Moreover, the additive decomposability is not the unique way to decompose entropy indices. Indeed, DEC2 may be adapted for the squared coefficient of variation (I_β, $\beta \to 1$), since it comprises all pairs of squared income differences.

6 Appendix

Proof. Theorem (3.1) :

(*Necessity*). The demonstration goes along the line the literature we find on distance measures (see Dagum (1980), Chakravarty and Dutta (1987), Ebert (1984)) or deprivation measures (Ebert and Moyes (2000), Bossert and D'Ambrosio (2006)). Consider there exists a measure g that satisfies INV, LIN, SM, NM and DEC1. Let $X \in \mathbb{R}^n_+$ and $Y \in \mathbb{R}^m_+$, for all $n, m \geq 2$ with $x_i \in X$, $x_r \in X$ and $y_j \in Y$. Suppose $x_i = x_r$, then $g(x_i, Y) = g(x_r, Y)$. Let us restrict the distribution Y to the j-th person. Then, there exists an elementary function $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that : $g = \phi(x_i, y_j) = \phi(x_r, y_j)$. By INV, we get : $\phi(x_i, y_j) = \phi(x_i + \delta, y_j + \delta)$. Let $\delta := -x_i$ such that $x_i > y_j$, this entails : $\phi(x_i, y_j) = \phi(0, x_i - y_j)$. As $\phi(\cdot)$ has to be defined over positive reals, if $x_i < y_j$, $\phi(x_i, y_j) = \phi(0, y_j - x_i)$. In both cases, if $\lambda := \frac{1}{x_i - y_j}$ or $\lambda := \frac{1}{y_j - x_i}$ by LIN we obtain : $\phi(x_i, y_j) = (y_j - x_i)\phi(0, 1)$ or $\phi(x_i, y_j) = (x_i - y_j)\phi(1, 0)$. By SM and NM, we get $\phi(0, 1) = \phi(1, 0) = 1$. Again by SM, for all $x_i < y_j$ or $x_i > y_j$, we obtain : $\phi(x_i, y_j) = |x_i - y_j|$, which is the well-known one-dimensional Hölderian distance function. From DEC1, we know that each x_i is compared with y_j for all $j \in \{1, 2, \dots, m\}$. Then, comparing x_i and y_j for all j together with NM gives : $g(x_i, Y) = \frac{\sum_{j=1}^m |x_i - y_j|}{m}$. Always from DEC1, repeating the comparisons for all $i \in \{1, 2, \dots, n\}$, together with NM : $g(X, Y) = \frac{\sum_{i=1}^n \sum_{j=1}^m |x_i - y_j|}{nm}$. Remark that from PP, it is easy to see that $h(X, Y) := \sum_{i=1}^n \sum_{j=1}^m |x_i - y_j|$ implies $h(X^{(t)}, Y^{(t)}) = t^2 \sum_{i=1}^n \sum_{j=1}^m |x_i - y_j|$, $\forall t \geq 2$. On the other hand, the normalization function $f(x) := \frac{x}{n}$ is relevant since after t concatenations, it is $f(x^{(t)}) = \frac{x}{t^2 n}$. Consequently, the axiom NM characterized by $f(\cdot)$ implies the respect of PP. The reverse is not true. Consider we normalize $h(X, Y)$ by $f(x) := \frac{x}{kn}$ with $k \in \mathbb{R}_+$. This normalization implies that PP is satisfied for all $k \in \mathbb{R}$ whereas NM is only available for $k = 1$. Then, only invoking NM, INV, LIN, SM, and DEC1 yields :

$$GMD(X, Y) = \frac{\sum_{i=1}^n \sum_{j=1}^m |x_i - y_j|}{nm}.$$

(*Sufficiency*). We can obviously verify that $GMD(\cdot)$ satisfies all axioms.

(*Independence*). In order to have a relevant *iff* axiomatic derivation, the axioms must be independent. We must find five different indices that satisfy all axioms but one (see Ebert and Moyes, 2000).

$$GMD^1(X, Y) = \frac{\sum_{i=1}^n \sum_{j=1}^m \sqrt{|x_i^2 - y_j^2|}}{nm} \qquad \text{(not INV)}$$

Remark that $GMD^1(X, Y)$ does not satisfy INV if, and only if, $(X, Y) \neq (\mathbb{1}_x, \mathbb{1}_y)$ and $(X, Y) \neq (\mathbb{O}_x, \mathbb{O}_y)$.

$$GMD^2(X, Y) = \frac{\sum_{i=1}^n \sum_{j=1}^m |x_i - y_j|^2}{nm} \qquad \text{(not LIN)}$$

$$GMD^3(X,Y) = \sum_{i=1}^{n}\sum_{j=1}^{m}|x_i - y_j| \qquad \text{(not NM)}$$

Remark that, since GMD^3 does not satisfy NM, it does not satisfy PP.

$$GMD^4(X,Y) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - y_j)}{nm} \qquad \text{(not SM)}$$

$$GMD^5(X,Y) = \frac{\sum_{i=1}^{n}|x_i - y_i|}{n} \qquad \text{(not DEC1)}$$

■

Proof. Theorem (3.2) :

(*Necessity*) Let us suppose there exists a measure of between-group inequality I_b satisfying SNM, SM, DEC2, and BPD. SNM is the intersection of NM and RS. As SNM $\Longrightarrow$ RS, Euler's functional equation for homogeneous functions applies (see Aczél (1966, p. 239)). Indeed from,

$$F(xz, yz) = z^k F(x,y), \forall z \neq 0,$$

follows the equations :

$$F(x \cdot 1, x \cdot \frac{y}{x}) = x^k F\left(1, \frac{y}{x}\right) = x^k g\left(\frac{y}{x}\right), \forall x \neq 0.$$

From SNM, we have $k = 0$ and from DEC2, it is possible to focus on two individuals y_j and x_i such that $I_b(x_i, y_j) = F\left(1, \frac{y_j}{x_i}\right), \forall x_i \neq 0$. Suppose $y_j = 0$, then from SNM we get $F(1,0) = 1$, which implies obvious solutions such that :

$$F(x_i, y_j) = \begin{cases} x_i + y_j \\ x_i - y_j, & \forall x_i = 1, y_j = 0. \\ x_i^{y_j} \end{cases}$$

From Euler's theorem for homogeneous functions of degree zero, we have :

$$\frac{\partial F(x_i, y_j)}{\partial x_i} = -\frac{\partial F(x_i, y_j)}{\partial y_j}. \qquad \text{(A1)}$$

Then, the function $F(x_i, y_j) = x_i - y_j$ holds. From SM, this reduces to :

$$F(x_i, y_j) = |x_i - y_j|, \forall x_i = 1, y_j = 0. \qquad \text{(A2)}$$

Invoking SNM again, we must have $F(x_i, 0) = 1, \forall x_i \neq 0$. This requires $F(x_i, y_j)$ being normalize in different ways. Remember that from SNM we have $Y = \mathbb{O}_y$, and $X = \mathbb{1}_x$ such that $F(eX, Y) = 1$ for all $e \in \mathbb{R}_{++}$. Always from SNM we can impose $x_i = \bar{x}$, where $\bar{x}$ is the arithmetic mean of X. Then, from SNM, four solutions consistent with (A1) and (A2) are :

$$F(x_i, y_j) = \begin{cases} \frac{|x_i - y_j|}{x_i + y_j} \\ \frac{|x_i - y_j|}{\bar{x} + \bar{y}} \\ \frac{|\bar{x} - \bar{y}|}{x_i + y_j} \\ \frac{|\bar{x} - \bar{y}|}{\bar{x} + \bar{y}} \end{cases}, \forall \bar{x}, \bar{y} \neq 0, \forall x_i, y_j \neq 0.$$

Now, invoking DEC2, we must be able to compute the contribution of $F(x_i, y_j)$ to $I_b(X, Y)$. Remember that DEC2 comprises $n \times m$ income pairs, then repeating all possible comparisons together with SNM yields :

$$C_F(x_i, y_j) = \begin{cases} \frac{|x_i - y_j|}{nm(x_i + y_j)} \\ \frac{|x_i - y_j|}{nm(\bar{x} + \bar{y})} \\ \frac{|\bar{x} - \bar{y}|}{nm(x_i + y_j)} \\ \frac{|\bar{x} - \bar{y}|}{nm(\bar{x} + \bar{y})} \end{cases}, \forall \bar{x}, \bar{y} \neq 0, \forall x_i, y_j \neq 0.$$

DEC2 together with SNM gives $I_b(X, Y) = \sum_{i,j} C_F(x_i, y_j)$, and then $f(x) = x$, implying the following class of inequality measures :

$$I_b\left(C_F(x_i, y_j)\right) = \begin{cases} \sum_{i,j} \frac{|x_i - y_j|}{nm(x_i + y_j)} \\ \sum_{i,j} \frac{|x_i - y_j|}{nm(\bar{x} + \bar{y})} \\ \sum_{i,j} \frac{|\bar{x} - \bar{y}|}{nm(x_i + y_j)} \\ \sum_{i,j} \frac{|\bar{x} - \bar{y}|}{nm(\bar{x} + \bar{y})} \end{cases}, \forall \bar{x}, \bar{y} \neq 0, \forall x_i, y_j \neq 0.$$

Imagine a progressive BPD transfer of amount $\delta > 0$ occurs from x_i to y_j $(x_i > y_i)$, such that $\bar{x} < \bar{y}$. As $C_{F_3}(x_i, y_j) = \frac{|\bar{x} - \bar{y}|}{x_i + y_j}$ and $C_{F_4}(x_i, y_j) = \frac{|\bar{x} - \bar{y}|}{\bar{x} + \bar{y}}$ violate BPD, we can restrict our attention to the other ones. Again from BPD, our class of inequality measures reduces to $I_b\left(C_{F_2}(X, Y)\right)$, such that $n = m$, that is, $\forall X, Y \in \Omega$. In consequence, invoking DEC2, BPD, SNM and NM yields :

$$G(X, Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{|x_i - y_j|}{n^2\left(\bar{x} + \bar{y}\right)}, \forall X, Y \in \Omega.$$

(*Sufficiency*) $G(X, Y)$ satisfies SNM, SM, DEC2, and BPD (for the latter, see the proof of Lemma (3.1) *infra*).

(*Independence*)

$$G^1(X, Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{|x_i - y_j|}{\bar{x} + \bar{y}} \qquad \text{(not SNM)}$$

$$G^2(X, Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{|x_i - y_j|}{n^2\left(\bar{x} - \bar{y}\right)}, \forall X > Y \qquad \text{(not SM)}$$

$$G^3(X,Y) = \sum_{i=1}^{n} \frac{|x_i - y_i|}{n\,(\bar{x}+\bar{y})} \qquad \text{(not DEC2)}$$

$$G^4(X,Y) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{|x_i - y_j|}{n^2\,(x_i+y_j)} \qquad \text{(not BPD)}$$

■

Proof. Lemma (3.1) : For the demonstration, one needs to introduce the concept of conditional rank. The conditional rank of an individual $r_{x_i/Y}$ is the rank of x_i if he belonged to Y. Let $r_{y_i/X}$ be the conditional rank of y_i in X.

(*Necessity* ($\imath$)). Let us decompose the numerator of $G(X,Y)$ in order to obtain the set $\mathcal{S}$ of all pair-based income differences :

$$\mathcal{S} = \left\{ \begin{array}{cccc} |x_1-y_1|, & |x_1-y_2| & ,\dots, & |x_1-y_m| \\ |x_2-y_1|, & |x_2-y_2| & ,\dots, & |x_2-y_m| \\ \vdots & \vdots & \vdots & \vdots \\ |x_n-y_1|, & |x_n-y_2| & ,\dots, & |x_n-y_m| \end{array} \right\}.$$

If a between-group Pigou-Dalton transfer occurs from $x_i \in X$ to $y_j \in Y$, then the set of pair-based income differences is :

$$\widetilde{\mathcal{S}} = \left\{ \begin{array}{ccccccccc} |x_i-\delta-y_1|, & |x_i-\delta-y_2| & ,\dots, & |x_i-y_j-2\delta| & ,\dots, & |x_i-\delta-y_{k+1}| & ,\dots, & |x_i-\delta-y_m|, \\ |y_j+\delta-x_1|, & |y_j+\delta-x_2| & ,\dots, & |y_j+\delta-x_\ell| & ,\dots, & |y_j+\delta-x_{\ell+1}| & ,\dots, & |y_j+\delta-x_n| \end{array} \right\}$$

$$= \left\{ \begin{array}{ccccccccc} |x_i-y_1|-\delta, & |x_i-y_2|-\delta & ,\dots, & |x_i-y_j|-2\delta & ,\dots, & |x_i-y_{k+1}|+\delta & ,\dots, & |x_i-y_m|+\delta, \\ |y_j-x_1|+\delta, & |y_j-x_2|+\delta & ,\dots, & |y_j-x_s|+\delta & ,\dots, & |y_j-x_{\ell+1}|-\delta & ,\dots, & |y_j-x_n|-\delta \end{array} \right\}.$$

For a BPD transfer from x_i to y_j with $r_{x_i/Y} = k$ and $r_{y_j/X} = \ell$, a simple computation gives the variation of the numerator of $G(X,Y)$: $\delta(2\ell-n+1)-\delta(2k-m+1)-2\delta$. The BPD transfer implies : $nm(\overline{\overline{x}}+\overline{\overline{y}}) = nm(\overline{x}-\frac{\delta}{n}+\overline{y}+\frac{\delta}{m}) > nm(\overline{x}+\overline{y})$. If $n=m$, and $k-\ell=-1$:

$$\begin{aligned} G(X,Y) &= \frac{\sum_{i=1}^{n}\sum_{r=1}^{m}|x_i-y_r|}{nm(\overline{x}+\overline{y})} \\ &= \frac{\sum_{i=1}^{n}\sum_{r=1}^{m}|x_i-y_r| + \delta(2\ell-n+1)-\delta(2k-m+1)-2\delta}{nm(\overline{x}+\overline{y})+nm(-\frac{\delta}{n}+\frac{\delta}{m})} =: G(\widetilde{X},\widetilde{Y}). \end{aligned}$$

If $n > m$, and $k-\ell > -1$:

$$\begin{aligned} G(X,Y) &= \frac{\sum_{i=1}^{n}\sum_{r=1}^{m}|x_i-y_r|}{nm(\overline{x}+\overline{y})} \\ &> \frac{\sum_{i=1}^{n}\sum_{r=1}^{m}|x_i-y_r| + \delta(2\ell-n+1)-\delta(2k-m+1)-2\delta}{nm(\overline{x}+\overline{y})+nm(-\frac{\delta}{n}+\frac{\delta}{m})} =: G(\widetilde{X},\widetilde{Y}). \end{aligned}$$

(*Sufficiency* ($\imath$)). Let $k-\ell < -1$ and $n < m \Longrightarrow G(\widetilde{X},\widetilde{Y}) > G(X,Y)$.

(ii) It can be noticed that a between-group Pigou-Dalton transfer from x_i to y_j, such that $x_i > y_j$ and $X, Y \in \Omega$, systematically implies $r_{x_i/Y} > r_{y_j/X}$. Hence, $G(\widetilde{X}, \widetilde{Y}) < G(X, Y)$. In consequence, for distributions with equal sizes, the restriction imposed on conditional ranks can be dropped and the respect of BPD is strict. ■

Proof. Lemma (3.2) :

Without loss of generality, suppose a within-group Pigou-Dalton transfer (WPD) occurring between $x_i, x_{i-1} \in X$. We then obtain the set of binary income differences :

$$\widetilde{S}^k = \left\{ \begin{array}{cccccc} |x_i - \delta - y_1|, & |x_i - \delta - y_2| & ,\ldots, & |x_i - \delta - y_j| & |x_i - \delta - y_{j+1}| & ,\ldots, & |x_i - \delta - y_m| \\ |x_{i-1} + \delta - y_1|, & |x_{i-1} + \delta - y_2| & ,\ldots, & |x_{i-1} + \delta - y_j| & |x_i - \delta - y_{j+1}| & ,\ldots, & |x_{i-1} + \delta - y_m| \end{array} \right\}$$

$$= \left\{ \begin{array}{cccccc} |x_i - y_1| + \delta, & |x_i - y_2| + \delta & ,\ldots, & |x_i - y_j| + \delta & |x_i - y_{j+1}| - \delta & ,\ldots, & |x_i - y_m| - \delta \\ |x_{i-1} - y_1| - \delta, & |x_{i-1} - y_2| - \delta & ,\ldots, & |x_{i-1} - y_j| - \delta & |x_i - y_{j+1}| + \delta & ,\ldots, & |x_{i-1} - y_m| + \delta \end{array} \right\}.$$

Imagine that $\exists y_j : x_{i-1} < y_j < x_i < y_{j+1}$, $j \in \{1, \ldots, m\}$. Then, the variation of the numerator of $G(X, Y)$ is -2δ (the denominator being constant). By induction, if b individuals are positioned in the ladder between the donor x_i and the receiver x_{i-1}, the variation is $-2b\delta$. The same reasoning applies when a WPD transfer occurs between y_i and y_{i-1} and if $\exists x_j : y_{i-1} < x_j < y_i$, $j \in \{1, \ldots, n\}$. Now consider both WPD transfers. As the WPD transfer preserves individuals' ranking, if there exists at least one person between the donor and the beneficiary (in X and in Y), then $G(\widetilde{X}, \widetilde{Y}) < G(X, Y)$.

Clearly, if $\nexists y_j : x_{i-1} < y_j < x_i$, $j \in \{1, \ldots, m\}$, then a WPD transfer from x_i to x_{i-1} implies that the number of $-\delta$ necessarily equals that of δ and, $G(\widetilde{X}, \widetilde{Y}) = G(X, Y)$.

The same reasoning applies for a transfer within Y (WPD1) or a transfer occurring both in X and in Y simultaneously (WPD2). ■

Proof. Theorem (3.3) :

The between-quantile transfers are characterized by doubly stochastic matrices. Any set of doubly stochastic matrices is a compact convex set, which is determined by the envelope of its extreme points. By Birkhoff's theorem (see Marshall and Olkin (1979, A.2, p. 19), extreme points of any doubly stochastic matrix are given by all permutation matrices. The doubly stochastic matrices B are expressed as :

$$B = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}, \alpha \in (0,1),$$

or by :

$$B = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \alpha \in (0,1).$$

Then, the convex hull is simply a line segment without extremities in $\mathbb{R}^4$. Remark that the B matrices are also denominated $T-$transforms (see Marshall and Olkin (1979, p.

21)). Obviously, the product $AB =: C$ provide a finite sequence of between-quantile transfers :

$$C = \begin{pmatrix} \alpha x_1 + (1-\alpha)y_1 & (1-\alpha)x_1 + \alpha y_1 \\ \vdots & \vdots \\ \alpha x_n + (1-\alpha)y_n & (1-\alpha)x_n + \alpha y_n \end{pmatrix}, \alpha \in (0,1).$$

We obtain $C = (\widetilde{X}, \widetilde{Y})$, for all $0.5 < \alpha < 1$ and $C = (\widetilde{Y}, \widetilde{X})$ for all $0 < \alpha < 0.5$. For $\alpha = 0.5$, we obtain identical distributions : $\widetilde{X} = \widetilde{Y}$. In both cases, the between-group Pigou-Dalton transfer is respected since : $\widetilde{x}_1 \leq \widetilde{x}_2 \leq \ldots \leq \widetilde{x}_n$ and $\widetilde{y}_1 \leq \widetilde{y}_2 \leq \ldots \leq \widetilde{y}_n$.

($\imath$). Between-quantile transfers from $x_i \in X$ to $y_i \in Y$ with $x_i \geq y_i$ imply $r_{x_i/Y} \geq r_{y_i/Y}$. Symmetrically, between-quantile transfers from $y_i \in Y$ to $x_i \in X$ $y_i \geq x_i$ imply $r_{y_i/X} \geq r_{x_i/X}$. Remember that $G(X,Y)$ is a symmetric function. Then, for all $\alpha \in (0,1)$, the desired result is provided by Lemma (3.1). For $\alpha = 0$ and $\alpha = 1$, $G(\widetilde{X}, \widetilde{Y}) = G(X,Y)$.

($\imath\imath$). Following Marshall and Olkin (1979, p. 430), for all $n \times d$ matrices $A, C \in \mathbb{R}^{nd}$, C is said to be *majorized* by A, that is $C \prec^d A$, if $C = AB$, where B is a $d \times d$ doubly stochastic matrix. Note that, C is said to be *chain majorized* by A, $C \preccurlyeq A$, if C is obtained by A by a finite product of $d \times d$ $T-$transforms. Any $T-$transform yields exactly a Pigou-Dalton transfer and *chain majorization* is equivalent to *majorization* if, and only if, $d = 2$ (see Marshall and Olkin (1979), p. 431). Note here that majorization is concerned with the rows of matrices A and C in the sense that each row of C (each rank or each quantile) is less spread out than the corresponding row of A.

($\imath\imath\imath$). To prove n-variate Lorenz dominance, we have to compute geometrical Lorenz figures in $\mathbb{R}^{d+1}$, that is, Lorenz *zonotopes* $LZ(\cdot)$, which are convex polytopes (see Koshevoy and Mosler (1996)). Let $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^{nd}_{+}$. $\mathbb{X}$ dominates $\mathbb{Y}$ in the sense of $d-$variate Lorenz dominance if $LZ(\mathbb{X}) \subset LZ(\mathbb{Y})$. In other terms, $\mathbb{X} \preceq^d_L \mathbb{Y}$ if the Lorenz zonotope of $\mathbb{X}$ is a subset of the Lorenz zonotope of $\mathbb{Y}$ in $\mathbb{R}^{d+1}$. As explained in Koshevoy and Mosler (2005), the Lorenz zonotope $LZ(\cdot)$ is equivalent to the lift zonotope $\widehat{Z}(\cdot)$ of the relative data, where relative data are computed as follows :

$$\widehat{\mathbb{X}} = \begin{pmatrix} \frac{x_{11}}{\bar{x}_1} & \cdots & \frac{x_{1d}}{\bar{x}_d} \\ \vdots & \vdots & \vdots \\ \frac{x_{n1}}{\bar{x}_1} & \cdots & \frac{x_{nd}}{\bar{x}_d} \end{pmatrix}, \ \bar{x}_d = \frac{1}{n}\sum_{i=1}^{n} x_{id}.$$

Let $\widehat{\mathbb{X}}_i$ be the i-th row of $\widehat{\mathbb{X}}$, then $LZ(\mathbb{X}) = \widehat{Z}(\widehat{\mathbb{X}}) = \frac{1}{n}\sum_{i=1}^{n}[(0, \mathbb{O}_d), (1, \widehat{\mathbb{X}}_i)]$, where $\frac{1}{n}\sum_{i=1}^{n}[(0, \mathbb{O}_d), (1, \widehat{\mathbb{X}}_i)]$ is a Minkowski sum. Following the between-quantile transfer, no implication can be found on the Lorenz zonotopes $LZ(C)$ and $LZ(A)$. On the contrary, between-quantile transfers systematically imply $LZ(C^T) \subset LZ(A^T)$ in $\mathbb{R}^{n+1}$. Let us compute $\widehat{A^T}$ and $\widehat{C^T}$:

$$\widehat{A^T} = \begin{pmatrix} \frac{x_1}{(x_1+y_1)/d} & \cdots & \frac{x_n}{(x_n+y_n)/d} \\ \frac{y_1}{(x_1+y_1)/d} & \cdots & \frac{y_n}{(x_n+y_n)/d} \end{pmatrix}, \ \widehat{C^T} = \begin{pmatrix} \frac{\alpha x_1+(1-\alpha)y_1}{(x_1+y_1)/d} & \cdots & \frac{\alpha x_n+(1-\alpha)y_n}{(x_n+y_n)/d} \\ \frac{(1-\alpha)x_1+\alpha y_1}{(x_1+y_1)/d} & \cdots & \frac{(1-\alpha)x_n+\alpha y_n}{(x_n+y_n)/d} \end{pmatrix}, d = 2.$$

If $\widehat{A_i^T}$ and $\widehat{C_i^T}$ stand respectively for the i-th row of $\widehat{A^T}$ and $\widehat{C^T}$, we have :

$$LZ(A^T) = \widehat{Z}(\widehat{A^T}) = \frac{1}{d}\sum_{i=1}^{d}[(0,\mathbb{O}_n),(1,\widehat{A_i^T})],$$

$$LZ(C^T) = \widehat{Z}(\widehat{C^T}) = \frac{1}{d}\sum_{i=1}^{d}[(0,\mathbb{O}_n),(1,\widehat{C_i^T})].$$

As the Minkowski sum yields the Lorenz zonotope by adding the sets points by points, and as the vector $(1,\widehat{C_i^T})$ is always closer to the main diagonal of the unit hypercube of $\mathbb{R}^{n+1}$ than the vector $(1,\widehat{A_i^T})$, then $LZ(C^T) \subset LZ(A^T)$ if, and only if, $\alpha \in (0,1)$. Consequently, $C^T \preceq_L^n A^T$. Equivalently, Koshevoy and Mosler (2005) advanced other properties for zonotope inclusions : if the matrices have the same rank ($rk(A^T) = rk(C^T)$), then $C^T \preceq_L^n A^T$ if each column C^{iT} dominates A^{iT}. Remark that the 2×2 doubly stochastic matrices, which may be permutation matrices (that is, for all $\alpha \in [0,1]$), only give weak inclusion, or equivalently, weak n-variate Lorenz dominance.

($\imath v$). See Marshall and Olkin (1979, p. 433). ■

Proof. Theorem (4.2) :

($\imath$). A transfer from $x_{ik} \in X_k \in \mathbb{R}^{n_k}$ to $x_{rj} \in X_j \in \mathbb{R}^{n_j}$ declines the overall Gini index $\widetilde{\mathcal{G}} < \mathcal{G}$ (from the well-known Pigou-Dalton property). The transfer implies a within-group variation, precisely, a transformation of the inequalities within Π_k and Π_j. These variations are computed *via* the following sets :

$$\begin{aligned}\widetilde{S}^k &= \{\ |x_{ik}-\delta-x_{1k}|,\ \ |x_{ik}-\delta-x_{2k}|\ \ ,\ldots,\ \ |x_{ik}-\delta-x_{ik}+\delta|,\ \ |x_{ik}-\delta-x_{i_{+1}k}|\ \ ,\ldots,\ \ |x_{ik}-\delta-x_{n_kk}|\ \}\\ &= \{\ |x_{ik}-x_{1k}|-\delta,\ \ |x_{ik}-x_{2k}|-\delta\ \ ,\ldots,\ \ |x_{ik}-x_{ik}|,\ \ |x_{ik}-x_{i_{+1}k}|+\delta\ \ ,\ldots,\ \ |x_{ik}-x_{n_kk}|+\delta\ \},\end{aligned}$$

$$\begin{aligned}\widetilde{S}^j &= \{\ |x_{rj}+\delta-x_{1j}|,\ \ |x_{rj}+\delta-x_{2j}|\ \ ,\ldots,\ \ |x_{rj}+\delta-x_{rj}-\delta|\ \ |x_{rj}+\delta-x_{r_{+1}j}|\ \ ,\ldots,\ \ |x_{rj}+\delta-x_{n_jj}|\ \}\\ &= \{\ |x_{rj}-x_{1j}|+\delta,\ \ |x_{rj}-x_{2j}|+\delta\ \ ,\ldots,\ \ |x_{rj}-x_{rj}|,\ \ |x_{rj}-x_{r_{+1}j}|-\delta\ \ ,\ldots,\ \ ,\ldots,\ \ |x_{rj}-x_{n_jj}|-\delta\ \}.\end{aligned}$$

We see that a necessary condition to obtain a number of $-\delta$ higher than a number of δ in each group is $r_{x_{ik}/X_k} > r_{x_{rj}/X_j} \Longrightarrow \widetilde{\mathcal{G}}_w < \mathcal{G}_w$.

Let us now analyze the implications of the between-group transfer on the other distributions. First, the implication of the income donor x_{ik} when he is compared with members of the other groups Π_h such that $x_{rh} \in X_h$, $h \in \{1,2,\ldots,K\} \setminus \{k\}$:

$$\begin{aligned}\widetilde{S}^{kh} &= \{\ |x_{ik}-\delta-x_{1h}|,\ \ |x_{ik}-\delta-x_{2h}|\ \ ,\ldots,\ \ |x_{ik}-\delta-x_{rh}|\ \ |x_{ik}-\delta-x_{r_{+1}h}|\ \ ,\ldots,\ \ |x_{ik}-\delta-x_{n_hh}|\ \}\\ &= \{\ |x_{ik}-x_{1h}|-\delta,\ \ |x_{ik}-x_{2h}|-\delta\ \ ,\ldots,\ \ |x_{ik}-x_{rh}|-\delta\ \ |x_{ik}-x_{r_{+1}h}|+\delta\ \ ,\ldots,\ \ |x_{ik}-x_{n_hh}|+\delta\ \},\end{aligned}$$

Second, the implication of the income beneficiary $x_{rj} \in X_j$ when he is compared with members of the other groups Π_h such that $x_{rh} \in X_h$, $h \in \{1,2,\ldots,K\} \setminus \{j\}$:

$$\widetilde{S}_{jh} = \{ \ |x_{rj} + \delta - x_{1h}|, \quad |x_{rj} + \delta - x_{2h}| \quad , \dots , \quad |x_{rj} + \delta - x_{rh} - \delta| \quad |x_{rj} + \delta - x_{r_{+1}h}| \quad , \dots , \quad |x_{rj} + \delta - x_{n_h h}| \ \}$$
$$= \{ \ |x_{rj} - x_{1h}| + \delta, \quad |x_{rj} - x_{2h}| + \delta \quad , \dots , \quad |x_{rj} - x_{rh}|, \quad |x_{rj} - x_{r_{+1}h}| - \delta \quad , \dots , \quad , \dots , \quad |x_{rj} - x_{n_h h}| - \delta \ \} .$$

A necessary condition to obtain more $-\delta$ than δ is : $r_{x_{ik}/X_h} > r_{x_{rj}/X_h}$, $\forall k \neq h, \forall j \neq h, h \in \{1, \dots, K\}$, that is, the donor possesses a higher conditional rank in each distribution than the receiver. This is systematically respected since $x_{ik} > x_{rj}$. Third, we must analyze the implication of the between-group transfer on the inequalities between Π_k and Π_j. From Lemma (3.1), the donor's conditional rank must be $r_{x_{ik}/X_j} \geq r_{x_{rj}/X_k} - 1$. It then follows that : $\widetilde{\mathcal{G}}_{gb} < \mathcal{G}_{gb}$.

($\imath\imath$). Analogous to ($\imath$). ■

Proof. Theorem (4.3) :

($\imath$) We know, for any given Pigou-Dalton transfer, that : $\widetilde{\mathcal{G}} < \mathcal{G}$. In consequence, if a standard Pigou-Dalton transfer occurs in Π_k and in Π_j : we have $\widetilde{\mathcal{G}}_w < \mathcal{G}_w$. And from Lemma (3.2), it follows that $\widetilde{\mathcal{G}}_{gb} < \mathcal{G}_{gb}$.

($\imath\imath$) Analogous to ($\imath$). ■

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