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Rank-Dependent Measures of Bi-Polarization and Marginal Tax Reforms

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Abstract

In this paper, we investigate a dual class of bi-polarization indices, namely rank-dependent bi-polarization indices. We show that these indices may be characterized with the generalized positional transfer sensitivity property. We find necessary and sufficient conditions in order to identify bi-polarization-reducing marginal tax reforms. Precisely, we propose inverse positional dominance criteria based on the comparison of bi-polarization concentration curves. An illustration is presented using the Jordanian Household Expenditure and Income Survey 2002/2003.

Keywords: Bi-polarization, Stochastic Dominance, Tax Reform.
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1 Introduction

The late Professor Berrebi has dedicated an important part of his academic career studying income inequality. Although the concept of polarization started to appear in the economic literature in the 1990's, pioneer works of Berrebi and Silber (1988) related to distances between income distributions and mainly Berrebi and Silber (1989) on the flatness of income distributions have been shown to be related with measures of bi-polarization, bi-polarization being an index of dispersion between a variable (say income) and a central tendency such as the median (refer e.g. to Silber, Deutsch and Hanoka, 2007). Since then, the literature has provided many analyzes of income polarization (see Esteban and Ray, 1994, Wolfson, 1994 and 1997, Wang and Tsui, 2000 and Duclos, Esteban and Ray, 2004 among others). Referring to Chakravarty et al. (2007), which proposed absolute measures of bi-polarization, we introduce a general class of rank-dependent absolute bi-polarization indices. One interesting aspect pointed out in the paper is the analysis of the impact of public policies on bi-polarization. Precisely, we propose a method to identify bi-polarization reducing tax reforms.

In their seminal investigations, Yitzhaki and Slemrod (1991) have highlighted the construction of welfare-improving indirect tax reforms for all social welfare functions respecting the Pigou-Dalton transfer principle. They show that this mechanism can be addressed by checking for non-intersecting concentration curves. Makdissi and Mussard (2008a, 2008b) have extended this work by defining higher order of concentration curves that are linked to positional transfer principles. In this paper, we extend our previous work by characterizing similar generalized transfer principles in the context of rank dependent bi-polarization. We then present the concept of bi-polarization concentration curves and show how they can be used to identify polarization-reducing indirect tax reforms.

The remainder of the paper is organized as follow. The next Section presents the mathematical notations. In Section 3, we develop well-known transfer principles for bi-polarization measurement and characterize the bi-polarization indices with respect to these principles. Section 4 is devoted to the definitions of bi-polarization concentration curves and how they can be used to identify bi-polarization-reducing indirect tax reforms. Section 5 provides an illustration based on the Jordanian Household Expenditure and Income Survey 2002/2003.

2 Notations

Let us define the environment on which we intend to obtain bi-polarization-reducing tax reforms. On the one hand, we consider the following absolute rank-dependent bi-polarization index à la Yaari (1987, 1988):

$$P(\Phi) = \int_0^1 |\Phi(p) - \Phi(0.5)| \, v(p) \, dp \tag{1}$$

where $\Phi(p) = \inf\left\{y^E : F\left(y^E\right) \geq p\right\}$ is the left inverse continuous c.d.f. (cumulative distribution function), y^E the equivalent income, $F\left(y^E\right)$ the c.d.f. of equivalent incomes, and $v(p) \geq 0$ the frequency distortion function weighting an individual at the p-th percentile of the distribution. The concept of equivalent incomes y^E has been introduced by King (1983). To account for the effect of different prices across households/individuals, King (1983) uses the utility function of a reference household as a basis for defining equivalent incomes. Let $U_\ell(y_\ell, q_\ell, t)$ represent the indirect utility of household ℓ, endowed with exogenous income y_ℓ, when facing prices q_ℓ and tax system t. Next, consider a *reference household* R that faces prices q_R. Accordingly, King (1983) defines the equivalent income by the exogenous income $y_{\ell,t}$ that would allow the reference household facing prices q_R and tax system t to reach utility $U_\ell(y_\ell, q_\ell, t)$:

$$U_R(y_{\ell,t}, q_R, t) = U_\ell(y_\ell, q_\ell, t)\,. \tag{2}$$

Thus, if t_1 and t_2 denote the pre-reform and post-reform tax systems then $y_{\ell,t_2} - y_{\ell,t_1}$ can be considered as a money measure of the welfare change for household ℓ of changing the tax system from t_1 to t_2.

Without loss of generality, we can restrict our attention to the class of bi-polarization functions for which $v(p) \geq 0$ for all $p \in [0,1]$. In this respect, if $\mathbb{R}_+$ denotes the set of nonnegative real numbers, our largest set of absolute bi-polarization indices is:

$$\Omega^1 := \left\{ P \in \mathbb{R}_+ \;\middle|\; \begin{array}{lll} v_-(p) & \geq 0 & \text{is continuous and differentiable} \\ & & \text{almost everywhere } \forall p \in [0, 0.5[\\ v_+(p) & \geq 0 & \text{is continuous and differentiable} \\ & & \text{almost everywhere } \forall p \in]0.5, 1] \end{array} \right\}$$

such that,

$$\begin{aligned} P(\Phi) &= \int_0^{0.5} (\Phi(0.5) - \Phi(p)) \, v_-(p) \, dp + \int_{0.5}^1 (\Phi(p) - \Phi(0.5)) \, v_+(p) \, dp \\ &=: P^-(\Phi) + P^+(\Phi). \end{aligned} \tag{3}$$

In the sequel, a function such as v, an income y_ℓ or an income distribution y will be indexed either by $-$ or by $+$ in order to define it either for all $p \in [0, 0.5[$ or for all $p \in]0.5, 1]$, respectively, that is, on the left-hand side of the median or on the right-hand side.[1]

[1]Note that for the derivatives of $v(\cdot)$ below, we will exclude $p = 0.5$ since the absolute value is continuous on $[0,1]$ but not derivable at 0.5.

3 Transfer Principles and Characterization

All indices in Ω^1 are said to satisfy Pen's (1971) Parade principle. It is worth mentioning that Pen's parade is usually concerned with the comparison of (inverse) c.d.f. only. An absolute parade is then provided.[2] Just note that the parade has to be made either before or after the median (or both):

Principle 3.1 Pen's Parade (1971). *If the curve* $\Phi_A(0.5) - \Phi_A(p) =: \mathbf{C}_A^-(p)$ *for all* $p \in [0, 0.5]$ *lies nowhere below the following* $\Phi_B(0.5) - \Phi_B(p) =: \mathbf{C}_B^-(p)$ *for all* $p \in [0, 0.5]$*, that is,* $\mathbf{C}_A^-(p)$ *weakly dominates* $\mathbf{C}_B^-(p)$*, and/or if* $\Phi_A(p) - \Phi_A(0.5) =: \mathbf{C}_A^+(p)$ *for all* $p \in [0.5, 1]$ *lies nowhere below* $\Phi_B(p) - \Phi_B(0.5) =: \mathbf{C}_B^+(p)$ *for all* $p \in [0.5, 1]$*, that is,* $\mathbf{C}_A^+(p)$ *weakly dominates* $\mathbf{C}_B^+(p)$*, then* $P(\Phi_A) \geq P(\Phi_B)$*.*

We now define subsets of Ω^1 that will be linked to higher order principles. Let $v_-^{(\ell)}(\cdot)$ and $v_+^{(\ell)}(\cdot)$ be the ℓ-th derivative of the $v(\cdot)$ function on $[0, 0.5[$ and $]0.5, 1]$ respectively, with $v^{(0)}(\cdot)$ being the function itself. Accordingly, we restrict our attention to the following class of bi-polarization functions that satisfy the Pigou-Dalton principle of transfer (Pigou (1912) and Dalton (1920)):

Principle 3.2 Pigou-Dalton (PD). *An income distribution* $\widetilde{y}^-$ $(\widetilde{y}^+)$*, whose left inverse cumulative distribution function is* $\widetilde{\Phi}^-$ $(\widetilde{\Phi}^+)$*, is obtained from the distribution* y^- (y^+) *of left inverse c.d.f.* Φ^- (Φ^+) *by a progressive Pigou-Dalton transfer on the left-hand side (right-hand side) of the median if a transfer of amount* $\delta > 0$ *occurs from* y_i^- (y_i^+) *to* y_j^- (y_j^+) *such as* $y_i^- > y_j^-$ $(y_i^+ > y_j^+)$*, letting the median and their positions unchanged:* $y_{i-1}^- \leq y_i^- - \delta$*,* $y_j^- + \delta \leq y_{j+1}^-$ $(y_{i-1}^+ \leq y_i^+ - \delta,\ y_j^+ + \delta \leq y_{j+1}^+)$*.*
A bi-polarization index weakly satisfies (PD) if

$$P^-(\widetilde{\Phi}^-) \geq P^-(\Phi^-)\ ,\ P^+(\widetilde{\Phi}^-) \geq P^+(\Phi^-)\ ,\ P(\widetilde{\Phi}) \geq P(\Phi)\ ,$$

($\imath$) when either the transfer occurs on the left-hand side of the median only ;
($\imath\imath$) or the transfer occurs on the right-hand side of the median only ;
($\imath\imath\imath$) or both transfers occur (on the left and on the right), respectively.

Lemma 3.1 *For all* $P(\Phi) \in \Omega^1$*, if* $P(\Phi)$ *weakly satisfies (PD), then:*
($\imath$) $v_-^{(1)}(p) \geq 0$ *for all* $p \in [0, 0.5[$
($\imath\imath$) $v_+^{(1)}(p) \leq 0$ *for all* $p \in]0.5, 1]$*.*

[2] Note that in the literature, absolute Lorenz ordering is related to distributions with equal means in order to compare distributions with same mean incomes. In the sequel, focus is put on distribution with same median.

Proof.

($\imath$) Let us use a discrete notation, that is, a population of n individuals and a rank-ordered equivalent income distribution $y^{+E} = \left(y_1^+, p_1; y_2^+, p_2; \ldots; y_n^+, p_n\right)$ with values in $\mathbb{R}_+$, p_i being rational numbers. Let $P^+(\cdot)$ being defined as follows:

$$P^+ = \sum_{i=1}^{n} \left(y_i^+ - y^M\right) v_+ \left(\frac{i}{n}\right),$$

where y^M stand for the median of the distribution. Assume a right-hand-side-Pigou-Dalton transfer valued to be $\delta > 0$ from individual i_n to individual $i_{n-1} > 0.5$ such as $i_n = i_{n-1} + \gamma$, $\gamma > 0$. If P^+ weakly respects the (PD) principle, then the bi-polarization index before transfer must be lower than after the transfer, that is, with a slight abuse of notation:

$$\begin{aligned}
&\left(y_{i_n}^+ - y^M\right) v_+ \left(\frac{i_n}{n}\right) + \left(y_{i_{n-1}}^+ - y^M\right) v_+ \left(\frac{i_n}{n}\right) \\
\leq\ &\left(y_{i_{n-1}}^+ + \delta - y^M\right) v_+ \left(\frac{i_{n-1}}{n}\right) + \left(y_{i_n}^+ - \delta - y^M\right) v_+ \left(\frac{i_n}{n}\right) \\
\Longleftrightarrow\ & v_+ \left(\frac{i_n}{n}\right) \delta \leq v_+ \left(\frac{i_{n-1}}{n}\right) \delta \\
\Longleftrightarrow\ & v_+ \left(\frac{i_{n-1}}{n} + \frac{\gamma}{n}\right) \leq v_+ \left(\frac{i_{n-1}}{n}\right).
\end{aligned}$$

Divide both sides by $\frac{\gamma}{n}$ and let $\gamma \to 0$, hence: $v_+^{(1)}(\cdot) \leq 0$.

($\imath\imath$) Imagine a left-hand-side-Pigou-Dalton transfer valued to be $\delta > 0$ from individual $i_2 < 0.5$ to individual i_1 such as $i_2 = i_1 + \gamma$, $\gamma > 0$:

$$\begin{aligned}
&\left(y^M - y_{i_1}^-\right) v_- \left(\frac{i_1}{n}\right) + \left(y^M - y_{i_2}^-\right) v_- \left(\frac{i_2}{n}\right) \\
\leq\ &\left(y^M - y_{i_1}^- - \delta\right) v_- \left(\frac{i_1}{n}\right) + \left(y^M - y_{i_2}^- + \delta\right) v_- \left(\frac{i_2}{n}\right) \\
\Longleftrightarrow\ & v_- \left(\frac{i_2}{n}\right) \delta \geq v_- \left(\frac{i_1}{n}\right) \delta \\
\Longleftrightarrow\ & v_- \left(\frac{i_1}{n} + \frac{\gamma}{n}\right) \geq v_- \left(\frac{i_1}{n}\right).
\end{aligned}$$

Divide both sides by $\frac{\gamma}{n}$ and let $\gamma \to 0$, thus $v_-^{(1)}(\cdot) \geq 0$.

■

It then follows that all indices $P(\cdot) \in \Omega^1$ satisfying (PD) are in the following set:

$$\Omega^2 := \left\{ P \in \Omega^1 \;\middle|\; \begin{array}{lll} v_-^{(1)}(p) & \geq 0 & \text{is continuous and differentiable} \\ & & \text{almost everywhere } \forall p \in [0, 0.5[\\ v_+^{(1)}(p) & \leq 0 & \text{is continuous and differentiable} \\ & & \text{almost everywhere } \forall p \in]0.5, 1] \end{array} \right\}.$$

Now, in order to provide more structure to our index, that is, to include more normative judgments about transfers between agents and how the individual behind the veil of ignorance may react according to a large spectrum of transfers, we require the following definition:

Definition 3.1 *The left (right) variation of bi-polarization induced by a progressive Pigou-Dalton transfer valued to be $\delta > 0$ from the person at rank $p + \gamma$, $\gamma > 0$ to the one at rank p is expressed as, respectively:*

$$\Delta_{p,\gamma} P^- (\delta, \Phi) := P^-(\widetilde{\Phi}) - P^-(\Phi), \; for\ all\ p \in [0, 0.5[$$
$$\Delta_{p,\gamma} P^+ (\delta, \Phi) := P^+(\widetilde{\Phi}) - P^+(\Phi), \; for\ all\ p \in]0.5, 1].$$

Principle 3.3 Principle of 1st-degree Positional Transfer Sensitivity (PTS1). *If $\widetilde{\Phi}$ is obtained by a left-hand side (right-hand side) Pigou-Dalton transfer valued to be $\delta > 0$ occurring from a higher-income person to a lower-income one, with a given proportion of the population between them, then it is more valuable when it takes place near the median rather than at the tails of the distribution, respectively:*

$$\Delta_{p,\gamma} P^- (\delta, \Phi) \leq \Delta_{p',\gamma} P^- (\delta, \Phi), \forall p' > p, \; for\ all\ p, p' \in [0, 0.5[\qquad \text{(PTS1}^-\text{)}$$
$$\Delta_{p,\gamma} P^+ (\delta, \Phi) \geq \Delta_{p',\gamma} P^+ (\delta, \Phi), \forall p' > p, \; for\ all\ p, p' \in]0.5, 1]. \qquad \text{(PTS1}^+\text{)}$$

A bi-polarization index weakly satisfies (PTS1) if

$$P(\widetilde{\Phi}) \geq P(\Phi) \; ,$$

($\imath$) *when the positional transfer sensitivity occurs on the left-hand side of the median ;*
($\imath\imath$) *or the positional transfer sensitivity occurs on the right-hand side of the median ;*
($\imath\imath\imath$) *or both positional transfers occur (on the left and on the right).*

Lemma 3.2 *For all $P(\Phi) \in \Omega^2$, if $P(\Phi)$ weakly satisfies (PTS1), then*
($\imath$) $v_-^{(2)}(p) \geq 0$ *for all* $p \in [0, 0.5[$
($\imath\imath$) $v_+^{(2)}(p) \geq 0$ *for all* $p \in]0.5, 1]$.

Proof.

($\imath$) Let us assume a right-hand side Pigou-Dalton transfer valued to be $\delta > 0$ at the lower part of the y^+ distribution: from individual i_h to individual i_{h-1} such as $i_h = i_{h-1} + \gamma > 0.5$, $\gamma > 0$ and another Pigou-Dalton transfer valued to be $\delta > 0$ at the upper part of the y^+ distribution: from individual i_n to individual i_{n-1} such as $i_n = i_{n-1} + \gamma > 0.5$, $\gamma > 0$. If P weakly respects the (PTS1$^+$) principle, then the bi-polarization variation resulting from the right-hand side (PD) transfer at the lower part of the distribution y^+ must be higher than that resulting from the transfer at the upper part of that distribution:

$$\begin{aligned}
&\left(y^+_{i_{h-1}} + \delta - y^M\right) v_+\left(\frac{i_{h-1}}{n}\right) + \left(y^+_{i_h} - \delta - y^M\right) v_+\left(\frac{i_h}{n}\right) \\
&- \left(y^+_{i_{h-1}} - y^M\right) v_+\left(\frac{i_{h-1}}{n}\right) - \left(y^+_{i_h} - y^M\right) v_+\left(\frac{i_h}{n}\right) \\
\geq\ & \left(y^+_{i_{n-1}} + \delta - y^M\right) v_+\left(\frac{i_{n-1}}{n}\right) + \left(y^+_{i_n} - \delta - y^M\right) v_+\left(\frac{i_n}{n}\right) \\
&- \left(y^+_{i_{n-1}} - y^M\right) v_+\left(\frac{i_{n-1}}{n}\right) - \left(y^+_{i_n} - y^M\right) v_+\left(\frac{i_n}{n}\right).
\end{aligned}$$

We get:

$$v_+\left(\frac{i_{h-1}}{n}\right) - v_+\left(\frac{i_h}{n}\right) \geq v_+\left(\frac{i_{n-1}}{n}\right) - v_+\left(\frac{i_n}{n}\right).$$

Then,

$$v_+\left(\frac{i_{h-1}}{n}\right) - v_+\left(\frac{i_{h-1}}{n} + \frac{\gamma}{n}\right) \geq v_+\left(\frac{i_{n-1}}{n}\right) - v_+\left(\frac{i_{n-1}}{n} + \frac{\gamma}{n}\right).$$

Divide both side by $\frac{\gamma}{n}$ and let $\gamma \to 0$, hence:

$$-v_+^{(1)}\left(\frac{i_{h-1}}{n}\right) \geq -v_+^{(1)}\left(\frac{i_{n-1}}{n}\right).$$

Suppose that $i_{n-1} = i_{h-1} + \beta$, such as $\beta > 0$. Therefore:

$$v_+^{(1)}\left(\frac{i_{h-1}}{n}\right) \leq v_+^{(1)}\left(\frac{i_{h-1}}{n} + \frac{\beta}{n}\right).$$

Divide both side by $\frac{\beta}{n}$ and let $\beta \to 0$, thus $v_+^{(2)}(\cdot) \geq 0$, for all $p \in]0.5, 1]$.

($\imath\imath$) The same reasoning applies. Let us now assume a left-hand side Pigou-Dalton transfer valued to be $\delta > 0$ at the lower part of the y^- distribution: from individual i_ℓ to individual $i_{\ell-1}$ such as $i_\ell = i_{\ell-1} + \gamma < 0.5$, $\gamma > 0$ and another Pigou-Dalton transfer valued to be $\delta > 0$ at the upper part of the y^- distribution: from individual i_u to individual i_{u-1} such as $i_u = i_{u-1} + \gamma < 0.5$, $\gamma > 0$. If P weakly respects the (PTS1$^-$) principle, then the bi-polarization variation resulting from the left-hand side (PD) transfer at the upper part of the distribution y^- must be higher than that resulting from the transfer at the lower part of that distribution:

$$\begin{aligned}
&\left(y^M - y^-_{i_{\ell-1}} - \delta\right) v_-\left(\frac{i_{\ell-1}}{n}\right) + \left(y^M - y^-_{i_\ell} + \delta\right) v_-\left(\frac{i_\ell}{n}\right) \\
&- \left(y^M - y^-_{i_{\ell-1}}\right) v_-\left(\frac{i_{\ell-1}}{n}\right) - \left(y^M - y^-_{i_\ell}\right) v_-\left(\frac{i_\ell}{n}\right) \\
\leq\ &\left(y^M - y^-_{i_{u-1}} - \delta\right) v_-\left(\frac{i_{u-1}}{n}\right) + \left(y^M - y^-_{i_u} + \delta\right) v_-\left(\frac{i_u}{n}\right) \\
&- \left(y^M - y^-_{i_{u-1}}\right) v_-\left(\frac{i_{u-1}}{n}\right) - \left(y^M - y^-_{i_u}\right) v_-\left(\frac{i_u}{n}\right).
\end{aligned}$$

It then follows that:

$$v_-\left(\frac{i_{\ell-1}}{n}\right) - v_-\left(\frac{i_\ell}{n}\right) \geq v_-\left(\frac{i_{u-1}}{n}\right) - v_-\left(\frac{i_u}{n}\right).$$

Thus,

$$v_-\left(\frac{i_{\ell-1}}{n}\right) - v_-\left(\frac{i_{\ell-1}}{n} + \frac{\gamma}{n}\right) \geq v_-\left(\frac{i_{u-1}}{n}\right) - v_-\left(\frac{i_{u-1}}{n} + \frac{\gamma}{n}\right).$$

Divide both side by $\frac{\gamma}{n}$ and let $\gamma \to 0$:

$$-v_-^{(1)}\left(\frac{i_{\ell-1}}{n}\right) \geq -v_-^{(1)}\left(\frac{i_{n-1}}{n}\right).$$

Suppose that $i_{n-1} = i_{\ell-1} + \tau$, such as $\tau > 0$:

$$v_-^{(1)}\left(\frac{i_{\ell-1}}{n}\right) \leq v_-^{(1)}\left(\frac{i_{\ell-1}}{n} + \frac{\tau}{n}\right).$$

Divide both side by $\frac{\tau}{n}$ and let $\tau \to 0$, thus $v_-^{(2)}(\cdot) \geq 0$, for all $p \in [0, 0.5[$. ■

In order to impose much more structure on the bi-polarization index let us expose the generalized Positional Principle of Transfer Sensitivity (see Aaberge, 2009), which is a generalization of Mehran's (1976) and Kakwani's (1980) princilpe of transfers (they introduced this principle building on Kolm's diminishing transfer principle, 1976, based on "utilitarian" social welfare functions). In welfare theory, we assume there is a positive welfare variation at the bottom of the distribution coupled with a negative welfare variation at the top, such as the overall welfare variation remains positive. Imagine the variation is lower and lower in taking (positive) variations of (positive) variations, and so on:

$$\Delta^2_{p,\Gamma^2} P(\delta, \Phi) := \Delta_{p+\gamma_2,\gamma_1} P(\delta, \Phi) - \Delta_{p,\gamma_1} P(\delta, \Phi), \tag{4}$$

where $\Gamma^2 = (\gamma_1, \gamma_2)$, $\gamma_i > 0$,

$$\vdots$$

$$\Delta^s_{p,\Gamma^s} P(\delta,\Phi) := \Delta^{s-1}_{p+\gamma_s,\Gamma^{s-1}} P(\delta,\Phi) - \Delta^{s-1}_{p,\Gamma^{s-1}} P(\delta,\Phi), \tag{5}$$

where $\Gamma^s = (\gamma_1, \gamma_2, \ldots, \gamma_s)$, $\gamma_i > 0$. If we now imagine that this generalized principle is applied to the left-hand side of the distribution and to the right-hand side, we get:

Principle 3.4 Principle of sth-degree Positional Transfer Sensitivity (PTSs). *If $\widetilde{\Phi}$ is issued from either a right-hand side (PTSs) implying that*

$$\Delta^s_{p,\Gamma^s} P^+(\delta,\Phi) \geq \Delta^s_{p',\Gamma^s} P^+(\delta,\Phi), \ \forall p' > p \tag{PTSs^+}$$

or/and a left-hand side (PTSs) implying that

$$\Delta^s_{p,\Gamma^s} P^-(\delta,\Phi) \leq \Delta^s_{p',\Gamma^s} P^-(\delta,\Phi), \ \forall p' > p \tag{PTSs^-}$$

then, a bi-polarization index $P(\Phi)$ satisfies this Principle when

$$P(\widetilde{\Phi}) \geq P(\Phi).$$

Lemma 3.3 *For all $P(\Phi) \in \Omega^2$ for which $v^{(\ell)}(\cdot)$ $\forall \ell \in \{1, 2, \ldots, s\}$ is continuous and differentiable over $[0, 0.5[\cup]0.5, 1]$ almost everywhere, if $P(\Phi)$ weakly satisfies (PTSs), then:*
($\imath$) $v_-^{(\ell+1)}(p) \geq 0$
($\imath\imath$) $(-1)^{\ell+1} v_+^{(\ell+1)}(p) \geq 0$.

Proof.
See the appendix. ■

In order to recap about our entire class of rank-dependent bi-polarization indices characterized by the generalized positional transfer sensitivity, we define the following set:

$$\Omega^s := \left\{ P \in \Omega^1 \left| \begin{array}{ll} v_-(p) & \text{is continuous and } s\text{-time differentiable almost everywhere} \\ & \forall p \in [0, 0.5[\text{ such as } v_-^{(\ell)}(p) \geq 0 \ \forall \ell = 1, 2, \ldots, s-1 \\ v_+(p) & \text{is continuous and } s\text{-time differentiable almost everywhere} \\ & \forall p \in]0.5, 1] \text{ such as } (-1)^\ell v_+^{(\ell)}(p) \geq 0 \ \forall \ell = 1, 2, \ldots, s-1 \end{array} \right. \right\}.$$

The use of Ω^s is crucial to match the ethical principles introduced *supra*. Finally, $P(\Phi) \in \Omega^1$ satisfies the Pen Parade Principle, $P(\Phi) \in \Omega^2 \subset \Omega^1$ also satisfies the Pigou-Dalton Principle of Transfers, $P(\Phi) \in \Omega^3 \subset \Omega^2 \subset \Omega^1$ also satisfies the Principle of 1st-degree Positional Transfer Sensitivity and $P(\Phi) \in \Omega^s \subset \Omega^{s-1} \subset \cdots \subset \Omega^3 \subset \Omega^2 \subset \Omega^1$ for all $s \in \{3, 4, \ldots\}$, also satisfies the Principle of $(s-2)$th-degree Positional Transfer Sensitivity. We add additional restrictions on Ω^s and define:

$$\widetilde{\Omega}^s := \left\{ P \in \Omega^s \left| \begin{array}{ll} v_-^{(\ell)}(0) = 0, & \forall \ell = 0, 1, 2, \ldots, s-1 \\ v_+^{(\ell)}(1) = 0, & \forall \ell = 0, 1, 2, \ldots, s-1 \end{array} \right. \right\}. \tag{6}$$

4 Bi-Polarization-Reducing Tax Reforms

This Section aims at gauging the impact of tax reforms à la Sandmo-Yitzhaki. Accordingly, the decision maker plans simultaneously a decreasing tax on commodity i and an increasing tax on commodity j, subject to a constant budget constraint. This marginal tax reform entails a variation in equivalent income $\Phi(p)$ for an individual at rank p:

$$d\Phi(p) = \frac{\partial \Phi(p)}{\partial t_i} dt_i + \frac{\partial \Phi(p)}{\partial t_j} dt_j. \tag{7}$$

Following Besley and Kanbur (1988) and Yitzhaki and Slemrod (1991), we use Roy's identity with the vector of reference prices sets to actual prices to assess the change in the equivalent income induced by a marginal change in the tax rate of good i. This change is:

$$\frac{\partial \Phi(p)}{\partial t_i} = -x_i(p), \tag{8}$$

where $x_i(p)$ is the Marshallian demand of good i of the individual at rank p in the income distribution. Let M be the number of goods, $m \in \{1, 2, \ldots, M\}$. Suppose a constant average tax revenue, $dR = 0$, where $R = \sum_{m=1}^{M} t_m X_m$ and where X_m is the average consumption of the m-th commodity: $X_m = \int_0^1 x_m(p)\, dp$. Yitzhaki and Slemrod (1991) prove that constant producer prices induce:

$$dt_j = -\alpha \left(\frac{X_i}{X_j} \right) dt_i \text{ where } \alpha = \frac{1 + \frac{1}{X_i} \sum_{m=1}^{M} t_m \frac{\partial X_m}{\partial t_i}}{1 + \frac{1}{X_j} \sum_{m=1}^{M} t_m \frac{\partial X_m}{\partial t_j}}. \tag{9}$$

Wildasin (1984) interprets α as the differential efficiency cost of raising one dollar of public funds by taxing the j-th commodity and using the proceeds to subsidize the i-th commodity. Substituting (9) and (8) in (7) yields:

$$d\Phi(p) = -\frac{x_i(p)}{X_i} X_i dt_i + \alpha \frac{x_j(p)}{X_j} X_i dt_i. \tag{10}$$

Let us now define the first-order bi-polarization concentration curve of good i as:

$$C_i^1(p) := \begin{cases} \frac{x_i(0.5) - x_i(p)}{X_i} & \text{for } p \in [0, 0.5[\\ \frac{x_i(p) - x_i(0.5)}{X_i} & \text{for } p \in]0.5, 1] \end{cases}. \tag{11}$$

It is simply the distance between the coordinate of the usual concentration curve of order 1 at point p, that is $x_i(p)/X_i$, to its coordinate at the median.[3] Integrating successively s times yields s-order bi-polarization concentration curves ($s-$curves for short):

$$C_i^s(p) := \begin{cases} C_i^s(p) = \int_p^{0.5} C_i^{s-1}(u) du & \text{for } p \in [0, 0.5[\\ C_i^s(p) = \int_{0.5}^{p} C_i^{s-1}(u) du & \text{for } p \in]0.5, 1] \end{cases}. \tag{12}$$

[3]The different order of concentration curves were introduced by Makdissi and Mussard (2008a, 2008b).

Therefore, the variation of bi-polarization induced by an indirect tax reform is:

$$dP(\Phi) = -X_i dt_i \int_0^1 \left[C_i^1(p) - \alpha C_j^1(p)\right] v(p)\, dp. \tag{13}$$

This leads to our first result:

Theorem 4.1 *An average-revenue-neutral marginal tax reform* $dt_j = -\alpha\left(\frac{X_i}{X_j}\right) dt_i > 0$ *implies* $dP(\Phi) \leq 0$ *for all* $P(\Phi) \in \Omega^s$ *if, and only if:*

$$C_i^s(p) - \alpha C_j^s(p) \leq 0 \ \forall p \in [0,1], s \in \{1, 2, 3, \ldots\}. \tag{14}$$

Proof.

(Sufficiency). The proof goes along the lines of Makdissi and Mussard (2008a) in the case of rank-dependent social welfare functions, but additional restrictions are needed.
∗ order $s = 1$: From (13), we immediately check that the condition holds since $v(p)$ is nonnegative and that dt_i is negative.

∗ order $s \in \{2, 3, \ldots\}$: Let first rewrite $\int_0^1 C_k^1(p)\, v(p)\, dp = \int_0^{0.5} C_k^1(p)\, v_-(p)\, dp + \int_{0.5}^1 C_k^1(p)\, v_+(p)\, dp$. Integrating $\int_0^{0.5} C_k^1(p)\, v_-(p)\, dp$ by parts for some $k \in \{i, j\}$ yields:

$$\int_0^{0.5} C_k^1(p)\, v_-(p)\, dp = -C_k^2(p)\, v_-(p)\Big|_0^{0.5} + \int_0^{0.5} C_k^2(p)\, v_-^{(1)}(p)\, dp. \tag{15}$$

By definition $v_-(0) = 0$ and $C_k^2(0.5) = 0$. This leads to

$$\int_0^{0.5} C_k^1(p)\, v_-(p)\, dp = \int_0^{0.5} C_k^2(p)\, v_-^{(1)}(p)\, dp. \tag{16}$$

Now, assume that for some $s > 2$, we have:

$$\int_0^{0.5} C_k^1(p)\, v_-(p)\, dp = \int_0^{0.5} C_k^{s-1}(p)\, v_-^{(s-2)}(p)\, dp. \tag{17}$$

Integrating by parts equation (17) and keeping in mind that $v_-^{(s-3)}(0) = 0$ and $C_k^{s-1}(0.5) = 0$, we get :

$$\int_0^{0.5} C_k^1(p)\, v_-(p)\, dp = \int_0^{0.5} C_k^s(p)\, v_-^{(s-1)}(p)\, dp. \tag{18}$$

Equation (16) respects the relation depicted in equation (17). We have shown that if equation (17) is true then equation (18) is also true. This implies that equation (18) is true for all integers $s \in \{2, 3, \ldots\}$.

Integrating now $\int_{0.5}^1 C_k^1(p)\, v_+(p)\, dp$ by parts for some $k \in \{i, j\}$ yields:

$$\int_{0.5}^1 C_k^1(p)\, v_+(p)\, dp = C_k^2(p)\, v_+(p)\Big|_{0.5}^1 - \int_{0.5}^1 C_k^2(p)\, v_+^{(1)}(p)\, dp. \tag{19}$$

By definition $v_+(1) = 0$ and $C_k^2(0.5) = 0$. This leads to

$$\int_{0.5}^{1} C_k^1(p)\, v_+(p)\, dp = \int_{0.5}^{1} C_k^2(p)\, v_+^{(1)}(p)\, dp. \tag{20}$$

Now, assume that for some $s > 2$, we have:

$$\int_{0.5}^{1} C_k^1(p)\, v_+(p)\, dp = (-1)^{s-2} \int_{0.5}^{1} C_k^{s-1}(p)\, v_+^{(s-2)}(p)\, dp. \tag{21}$$

Integrating by parts equation (21) and keeping in mind that $v_+^{(s-3)}(1) = 0$ and $C_k^{s-1}(0.5) = 0$, we get :

$$\int_{0.5}^{1} C_k^1(p)\, v_+(p)\, dp = \int_{0.5}^{1} C_k^{s}(p)\, v_+^{(s-1)}(p)\, dp. \tag{22}$$

Equation (20) respects the relation depicted in equation (21). We have shown that if equation (21) is true then equation (22) is also true. This implies that equation (22) is true for all integers $s \in \{2, 3, \ldots\}$.

From equations (13), (18) and (22), we obtain for $s \in \{2, 3, \ldots\}$:

$$\begin{aligned} dP(\Phi) \;&=\; -X_i dt_i \left\{ \int_{0}^{0.5} \left[C_i^s(p) - \alpha C_j^s(p) \right] v_-^{(s-1)}(p)\, dp \right. \\ &\qquad \left. + (-1)^{s-1} \int_{0.5}^{1} \left[C_i^s(p) - \alpha C_j^s(p) \right] v_+^{(s-1)}(p)\, dp \right\}. \end{aligned} \tag{23}$$

From the set Ω^s, remember that for $p \in [0, 0.5[$, $v_-^{(\ell)}(p) \geq 0$ and that for $p \in]0.5, 1]$, $(-1)^{\ell} v_+^{(\ell)}(p) \geq 0$ $\forall \ell \in \{1, 2, \ldots, s-1\}$ and that $dt_i < 0$. Therefore, a sufficient condition for $dP(\Phi) \leq 0$ is $C_i^s(p) - \alpha C_j^s(p) \leq 0$ for all $p \in [0, 1]$.

(Necessity). Consider the set of functions $P(\Phi) \in \widetilde{\Omega}^s$ for which the $(s-2)$th derivative of $v_+(p)$ is constant and the $(s-2)$th derivative of $v_-(p)$ is of the following form:

$$v_-^{(s-2)}(p) = \begin{cases} 0 & p \leq \overline{p} \\ p - \overline{p} & \overline{p} < p \leq \overline{p} + \epsilon \\ \epsilon & \overline{p} + \epsilon < p < 0.5 \end{cases}, \tag{24}$$

for some $\overline{p} \in [0, 0.5[$. Since $v_-(p)$ is differentiable almost everywhere except at $\overline{p}$ and $\overline{p} + \epsilon$, it satisfies the conditions in (6). Thus, bi-polarization indices whose frequency distortion functions $v_-(p)$ have the particular above form for $v_-^{(s-2)}(p)$ belong to $\widetilde{\Omega}^s$. This yields $v_+^{(s-1)}(p) = 0$ and:

$$v_-^{(s-1)}(p) = \begin{cases} 0 & p \leq \overline{p} \\ 1 & \overline{p} < p \leq \overline{p} + \epsilon \\ 0 & p > \overline{p} + \epsilon \end{cases}. \tag{25}$$

Imagine now that $C_i^s(p) - \alpha C_j^s(p) > 0$ on an interval $[\overline{p}, \overline{p} + \epsilon]$ for ϵ that can be arbitrarily close to 0. For $v_-(p)$ defined as in (24), expression (23) is then positive and the marginal tax reform induces a marginal increase in bi-polarization.

Consider now the set of functions $P(\Phi) \in \widetilde{\Omega}^s$ for which the $(s-2)$th derivative of $v_-(p)$ is constant and the $(s-2)$th derivative of $v_+(p)$ is such that

$$v_+^{(s-1)}(p) = \begin{cases} (-1)^{s-1}\epsilon & 0.5 < p \leq \overline{p} \\ (-1)^{s-1}(\overline{p}+\epsilon-p) & \overline{p} < p \leq \overline{p}+\epsilon \\ 0 & p > \overline{p}+\epsilon \end{cases}, \tag{26}$$

for some $\overline{p} \in]0.5, 1]$. Since $v_+(p)$ is differentiable almost everywhere except at $\overline{p}$ and $\overline{p}+\epsilon$, it satisfies the conditions in (6). Thus, polarization indices whose frequency distortion functions $v_+(p)$ have the particular above form for $v_+^{(s-1)}(p)$ belong to $\widetilde{\Omega}^s$. This yields $v_-^{(s-1)}(p) = 0$ and:

$$v_+^{(s)}(p) = \begin{cases} 0 & p \leq \overline{p} \\ (-1)^s & \overline{p} < p \leq \overline{p}+\epsilon \\ 0 & p > \overline{p}+\epsilon \end{cases}. \tag{27}$$

Imagine now that $C_i^s(p) - \alpha C_j^s(p) > 0$ on an interval $[\overline{p}, \overline{p}+\epsilon]$ for ϵ that can be arbitrarily close to 0. For $v_+(p)$ defined as in (26), expression (23) is then positive and the marginal tax reform induces a marginal increase in bi-polarization. Hence, it cannot be that $C_i^s(p) - \alpha C_j^s(p) > 0$ for $p \in [\overline{p}, \overline{p}+\epsilon]$.

■

This result states that a marginal tax reform increasing the tax on the j-th good and decreasing the tax on the i-th good produces a decrease of bi-polarization if the bi-polarization curve of order s of good j (multiplied by α) lies nowhere below that of good i. Furthermore, this test enables bi-polarization-reducing tax reform to be implemented in being aware of the behavior of the decision maker since each order s corresponds to a precise ethical transfer principle. Indeed, as far as s increases, the decision maker is more and more averse to rank dependent bi-polarization.

5 Application

In this section we perform double (left hand side plus right hand side) inverse stochastic dominance tests in order to identify marginal tax reforms for which the reduction of polarization is possible. We use the Jordanian Household Expenditure and Income Survey 2002/2003. Using a sample of 9,999 households, we investigate many commodity marginal tax reforms. The pairs of commodity on which tax polices are applied are the following: expenses in transport and communication *versus* education and transport (and communication) *versus* medical care.

In order to capture simple marginal tax reforms, we use $\alpha = 1$, that is a reform associated with neither efficiency gain nor efficiency loss for the government, that is, the ratio between each marginal social cost of funds is valued to be one.

The boundary of α is crutial. As Yitzhaki and Slemrod (1991) pointed out, $\alpha < 1$ ($\alpha > 1$) indicates, as a consequence of the tax reform, whether a diminution (a rise) of the excess burden occurs. Referring to Duclos, Makdissi and Wodon (2008) in the case of welfare-improving tax reforms, a wide range of efficiency parameters are operational. Nevertheless, in general, welfare indices being rank-dependent cannot be neither Pen improving, nor Dalton improving, nor Positional improving (for all orders) if $\alpha > 1$. Then, welfare-improving tax reforms are usually associated with $\alpha \leq 1$. The same result holds for bi-polarization.

In Figure 1, we expose a second-order inverse dominance test between transport and education. As can be seen, this second-order test does not allow for demonstrating that bi-polarization indices $P(\cdot) \in \Omega^2$ decreases when the decision maker marginally increases the tax on transport (and communication) and uses the proceed to decrease the tax on education. This is because the $s-$curves of order 2 cross on the right-hand side of the median.

[Figure 1]

On the contrary, if we suppose that the decision maker is more averse to bi-polarization, e.g, we make the same test at the order 3, then the $s-$curves do not cross as depicted in Figure 2. In this respect, if $s = 3$ bi-polarization decreases when the decision maker marginally decreases the tax on education, this tax being financed by a increasing tax on transport (and communication), the budget neutrality being respected.

[Figure 2]

The same reasoning applies for transport (increasing tax) and medical care (decreasing tax): Figures 3 and 4.

[Figure 3]

[Figure 4]

Appendix

Proof of Lemma 3.3:

From lemma 3.2, (PST1$^-$) implies $v_-^{(2)}(p) \geq 0$ and (PST1$^+$) implies $v_+^{(2)}(p) \geq 0$. We just show that (PST2$^-$) implies $v_-^{(3)}(p) \geq 0$ since higher order may be obtained using an induction reasonning.

Imagine a (PD) transfer in the tail of y^- valued to be δ from $y^-_{i_{\ell-1}}$ to $y^-_{i_{\ell-2}}$ coupled with another (PD) transfer valued to be δ from $y^-_{i_\ell}$ to $y^-_{i_{\ell+1}}$, namely a favorable composite transfer(since it decreases polarization). We compare this favorable composite transfer with another one near the median (valued to be δ) from $y^-_{i_{u-1}}$ to $y^-_{i_{u-2}}$ and from $y^-_{i_u}$ to $y^-_{i_{u+1}}$. From lemma 3.1, we get:

$$-v\left(\frac{i_{\ell-2}}{n}\right)+v\left(\frac{i_{\ell-1}}{n}\right)+v\left(\frac{i_{\ell}}{n}\right)-v\left(\frac{i_{\ell+1}}{n}\right) \geq -v\left(\frac{i_{u-2}}{n}\right)+v\left(\frac{i_{u-1}}{n}\right)+v\left(\frac{i_{u}}{n}\right)-v\left(\frac{i_{u+1}}{n}\right)$$
$$-v_-^{(1)}\left(\frac{i_{\ell-2}}{n}\right)-v_-^{(1)}\left(\frac{i_{\ell-2}}{n}\right) \geq v_-^{(1)}\left(\frac{i_{u-2}}{n}\right)-v_-^{(1)}\left(\frac{i_{u}}{n}\right)$$
$$-v_-^{(2)}\left(\frac{i_{\ell-2}}{n}\right) \geq -v_-^{(2)}\left(\frac{i_{u-2}}{n}\right).$$

Let $i_{\ell-2}+\tau = i_{u-2}$:

$$-v_-^{(2)}\left(\frac{i_{\ell-2}}{n}\right)+v_-^{(2)}\left(\frac{i_{\ell-2}+\tau}{n}\right) \geq 0.$$

Divide both sides of the last expression by $\frac{\tau}{n}$ and let $\tau \to 0$:

$$v_-^{(3)}(\cdot) \geq 0.$$

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Figure 1: Transport / Education - order 2

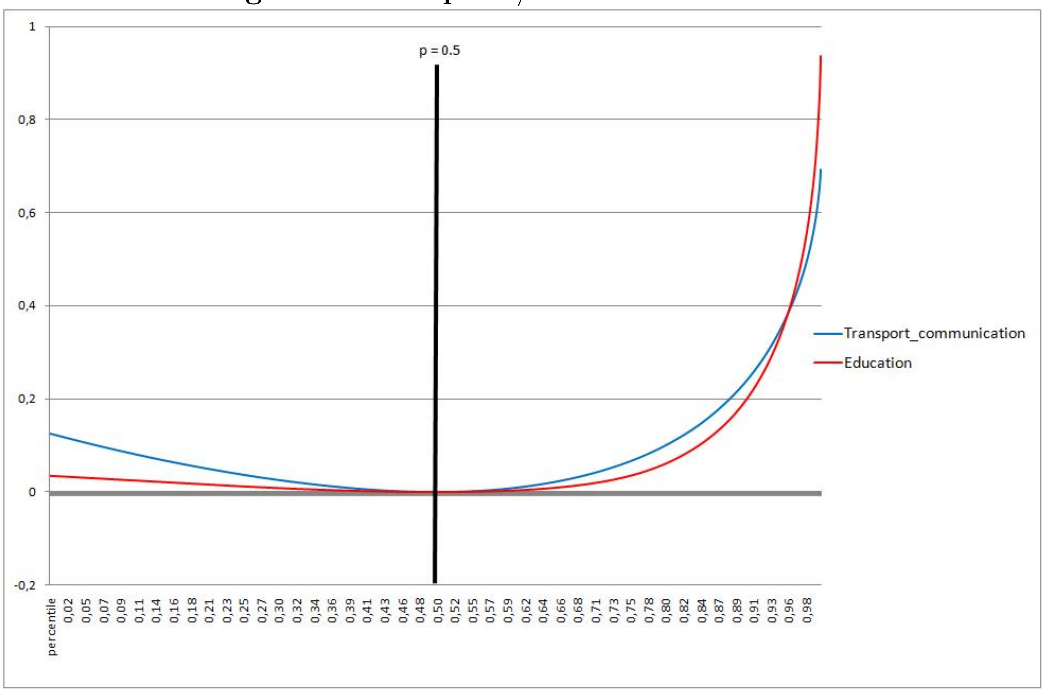

Figure 2: Transport / Education - order 3

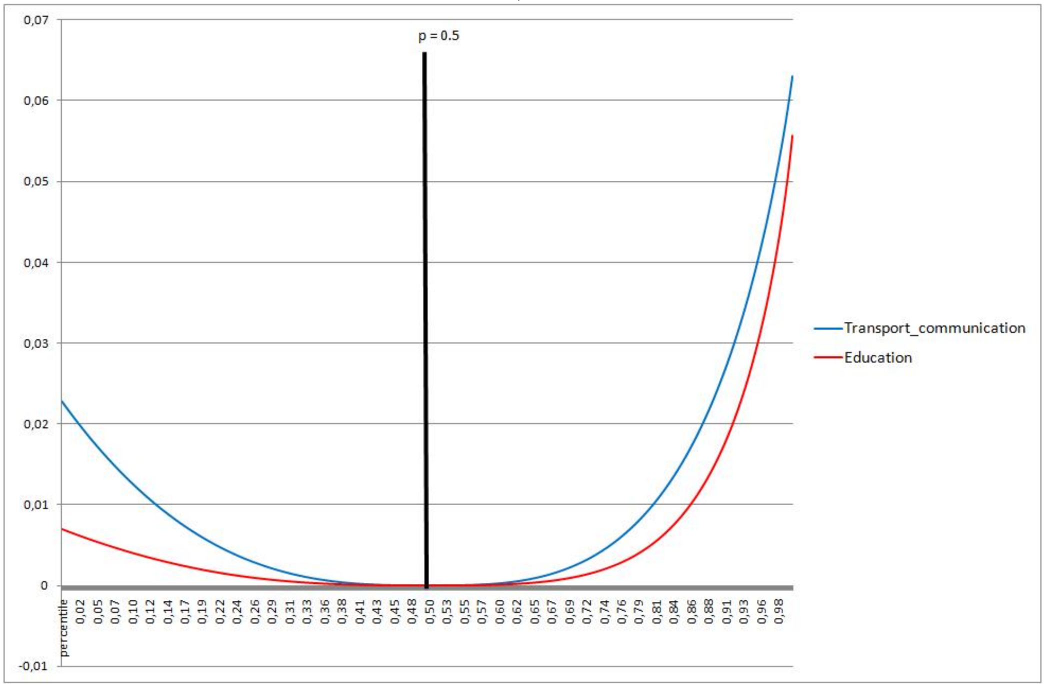

Figure 3: Transport / Medical care - order 2

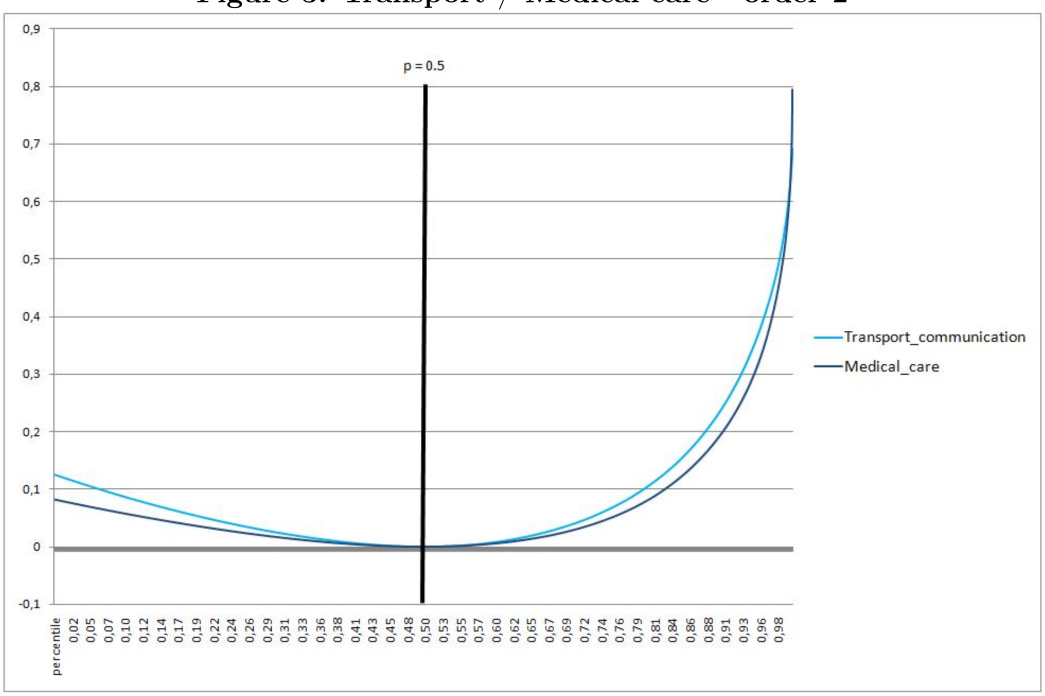

Figure 4: Transport / Medical care - order 3