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Mickael Beaud

Stéphane Mussard

UNIVERSITÉ DE
SHERBROOKE

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Mickael Beaud*
LAMETA
Université Montpellier I

Stéphane Mussard†
LAMETA
Université Montpellier I

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Abstract

In most welfare analyses, especially in the literature on normative inequality measurement, it is a commonplace to assume a direct relationship between the distribution of income and social welfare. As a result, this relationship is formally summarized by a single function, called a reduced-form social welfare function. Hence, with reference to some transfer principles, normative considerations are introduced and the shape of the reduced-form is accordingly restricted. In this note, we investigate and question the relevance of this approach. After recognizing that any reduced-form social welfare function merges two elements of very different nature – individuals' self-interested preferences over income (an empirical element) and those of the society over utilities (a normative element) – it is clear that it is problematic or at least misleading to make assumptions about the shape of the reduced-form social welfare function according to normative considerations only. However, we show that consistency can be restored whenever individuals' preferences can be represented by completely monotone utility functions.

Classification: D3; D6; H2.

Keywords: Social Welfare; Inequality; Transfer Principles; Completely Monotone Utility Functions.

*Address: Faculté d'Economie, Av. Raymond Dugrand, Site de Richter C.S. 79606, 34960 Montpellier Cedex 2, France. E-mail: beaud@lameta.univ-montp1.fr.

†Address: Faculté d'Economie, Av. Raymond Dugrand, Site de Richter C.S. 79606, 34960 Montpellier Cedex 2, France. E-mail: mussard@lameta.univ-montp1.fr.

1 Introduction

The distribution of income among individuals is undeniably one of the fundamental data upon which any regime, social state or public policy should be evaluated. Of course, economists have long been recognized that it does not tell us the whole story about the welfare of the society. As originally pointed out by Hugh Dalton (1920, p. 348):

> *"For the economist is primarily interested, not in the distribution of income as such, but in the effects of the distribution of income upon the distribution and total amount of economic welfare, which may be derived from income. We have to deal, therefore, not merely with one variable, but with two, or possibly more, between which certain functional relations may be presumed to exist".*

This claim is unequivocal by recognizing a nonlinear relationship between the distribution of income and social welfare. In most theoretical and applied welfare analyses, especially in the literature on normative inequality measurement, it is standard practice to summarize this relationship by a *reduced-form social welfare function* assumed to be symmetric and additively separable. In the case of a finite population of I individuals, letting $y^i \in \mathbb{R}_+$ be the income of the ith individual in the society (being in the non-negative real space), the reduced-form social welfare function may be expressed as follows:

$$\sum_{i=1}^{I} W\left(y^i\right). \tag{1}$$

This model constitutes a simple framework for the purpose of the theory, that is, to rank income distributions according to clearly specified *transfer principles.* As will be made clear hereafter, whether or not the reduced-form social welfare function in (1) meets some transfer principles typically depends on assumptions regarding the signs of the successive derivatives of W – this result is due to Fishburn and Willig (1984).

The purpose of this note is to investigate and to question the relevance of this approach. Our starting point is the recognition that the reduced-form social welfare function in (1) is a composite of $1+I$ functions: a symmetric generalized utilitarian social welfare function, $\sum_{i=1}^{I} V\left(U^i\right)$, where U^i is the utility level of the ith individual, and I identical Bernoulli utility functions over income, $U\left(y^i\right)$. Accordingly, (1) may also be written as a *function composition*:

$$\sum_{i=1}^{I} V\left(U\left(y^i\right)\right), \tag{2}$$

where

$$W\left(y^i\right) = V\left(U\left(y^i\right)\right) \tag{3}$$

implicitly holds.

A possible reaction to this is to deny the relevance of the distinction between U and V. It could be argued that, because the choice of a social welfare function involves choosing a particular cardinalization of individuals' utility functions, the choice of both U and V involves value judgments and are part of the same process[1]. There is, however, a strong argument in favor of keeping the distinction between U and V clear. The point is that the shape of individuals' Bernoulli utility functions is an empirical phenomenon which as such cannot be determined by reference to value judgments and, therefore, to transfer principles.

In principle, assumptions about the shape of U should be expected to follow from empirical observations, and it is only in the case where empirical results are lacking or non-unanimous that the analyst's subjective judgment could matter. However, the subjective judgment of a fact is definitely not a value judgment. In any event, U has no normative content and the issue of its plausible or acceptable properties belongs to positive economic analysis. On the other hand, the question of the appropriate shape of V belongs to normative economic analysis and should be expected to be grounded in a theory of distributive justice[2].

In most of the literature, it is commonplace to postulate that W is itself a Bernoulli utility function assumed to be the same for all individuals. The reduced-form social welfare function in (1) is then either interpreted as an utilitarian social welfare function or, after being multiplied by $\frac{1}{I}$, as the Von Neumann-Morgenstern expected utility function of any individual behind the veil of ignorance[3]. In both cases, the explicit value judgment involved is that any welfarist social planner (i.e. one being concerned with individual utility levels only), should be neutral to inequality in individuals' utility levels. Formally, normative considerations imply that V is linear in individual utility levels and, hence, whether or not the reduced-form social welfare function in (1) meets any particular transfer principle becomes an empirical question. Furthermore, the analyst may then be constrained to inconsistently link transfer principles (embodying value judgments) to the shape of individuals' Bernoulli utility functions (a purely empirical phenomenon).

In the utilitarian framework, inconsistency may arise because, once particular normative considerations have been introduced through the choice of a linear social welfare function, the analyst has to make assumptions about the shape of individuals' Bernoulli utility functions

[1] We owe this remark to Le Grand (1984, p. 243, footnote 1).

[2] The shape of V may also be viewed as an empirical phenomenon, as the inverse optimum tax problem illustrates (e.g. Ahmad and Stern, 1984). In a sense, solving this problem allows one to discover the theory of justice endorsed by the society (supposed to act optimally according to this theory). But, once again, the subjective judgment of a fact is not a value judgment, and even if the society's preferences are also viewed as an empirical phenomenon, it is one of very different nature than that of individuals' self-interested preferences.

[3] The normative contain of this approach is that income distributions should be ranked according to the preferences of "*a person taking a positive sympathetic interest in the welfare of each participant but having no partial bias in favor of any participant*" (Harsanyi, 1977, p. 49).

in order to get a reduced-form social welfare function satisfying some transfer principles. But once the social welfare function is fixed, there is no means for the analyst to introduce additional normative considerations.

On the other hand, whenever the reduced-form social welfare function is interpreted as a Von Neumann-Morgenstern expected utility function, the problem of the social ranking of income distributions becomes formally equivalent to that of individuals' choice under risk (this may help to explain the popularity of the approach). Indeed, behind the veil of ignorance, any individual ranks income distributions as lotteries. But this equivalence is only formal. Clearly, an individual may be a gambler and, at the same time, be conducive to income redistribution (whether or not she knows her position in the income distribution). Thus, the main failure of the expected utility framework is that it cannot allow for risk aversion and inequality aversion to be meaningfully distinguished.

Sometimes, however, it is recognized (Fleurbaey and Michel, 2001, p. 3) that:

> "... *the function W is best interpreted not as representing individual utility but, in a more general fashion, as embodying the way in which individual y^is are weighted in the social objective*".

But then, the question of what determines the shape of W is still left unclear. To our knowledge, despite recent and notable contributions by Louis Kaplow, the literature seems rather silent about the interplay between the functions U, V and W. In Kaplow (2010), it is shown that the degree of concavity of U and the degree concavity of V have asymmetric effects on the degree of concavity of W (see also Kaplow, 2008, and Kaplow and Weisbach, 2011). In particular, the more concave is U, the less relevant the concavity of V will be. The reason is that when U is more concave, if individuals' incomes are high, there may be significant differences in individuals' incomes and, at the same time, small differences in utility levels. In this case, even a high degree of concavity of V may not significantly increase social preferences for income redistribution.

In this note, we investigate the related, but different, issue of the precise relationship of the signs of the successive derivatives of W to those of both U and V. In Section 2, we briefly expose the standard approach of linking the shape of W to transfer principles. Hence, we reconsider the problem by keeping the distinction between U and V. We show that there is no trivial relationship between the signs of the successive derivatives of W and the ones of U and V. Then, we establish a sufficient condition that ensures that the reduced-form social welfare function meets many particular transfer principles. Finally, Section 3 summarizes our contribution and discusses some of its implications.

2 Generalized transfers principle and the shape of U, V and W

Among transfer principles, the weaker is known as *Pen's parade principle* (see Pen, 1971): the reduced-form social welfare function should be anonymous and monotone. Anonymity is simply a property of equal treatment of individuals, and monotonicity implies that W should positively reflect an increase in the income of any individual. Clearly, the reduced-form social welfare function in (1) satisfies this first-order transfer principle whenever the marginal social utility of income is non-negative:

$$W^{(1)}\left(y^i\right) := \frac{\partial W\left(y^i\right)}{\partial y^i} \geq 0, \quad \forall y^i \in \mathbb{R}_+. \tag{4}$$

The second-order transfer principle, originally discussed by Pigou (1912) and subsequently developed by Dalton (1920), is the so-called *Pigou-Dalton transfer principle.* This principle may be viewed as the cornerstone of the literature on normative inequality measurement. It states that any rank-preserving progressive transfer (i.e. a transfer form any individual to a poorer that do not affect individuals' rank within the income distribution) should reduce any inequality measure and, by duality, should be viewed as a social welfare improvement. The Pigou-Dalton transfer principle is satisfied whenever the marginal social utility of income is non-increasing:

$$W^{(2)}\left(y^i\right) := \frac{\partial W^{(1)}\left(y^i\right)}{\partial y^i} \leq 0, \quad \forall y^i \in \mathbb{R}_+. \tag{5}$$

Going a step further, Kolm (1976) introduced a third-order transfer principle called the *diminishing transfer principle*: a rank-preserving progressive transfer between individuals with fixed income difference is judged more desirable if these incomes are lower than if they are higher. The diminishing transfer principle is then satisfied whenever the marginal social utility of income is convex:

$$W^{(3)}\left(y^i\right) := \frac{\partial W^{(2)}\left(y^i\right)}{\partial y^i} \geq 0, \quad \forall y^i \in \mathbb{R}_+. \tag{6}$$

Subsequently Fishburn and Willing (1984) generalized this approach by considering s-order transfers principle ($s = 2, 3, \ldots$), the so-called *generalized transfers principle.* For $s \geq 2$, the idea is to compare a transfer of order $s-1$ among some individuals with another transfer of order $s-1$ among richer individuals. The implication of such generalized transfers is that we constrain the sign of the sth derivative of W. Accordingly, letting $\mathscr{C}^s$ be the set of all s-time differentiable functions, it is convenient to define

$$\Gamma^s := \left\{ f \in \mathscr{C}^s \;\middle|\; (-1)^s f^{(s)}(x) := (-1)^s \frac{\partial f^{(s-1)}(x)}{\partial x} \leq 0, \quad \forall x \in \mathbb{R} \right\}$$

as the class of real-valued functions which share the property of having their sth derivative non-positive (non-negative) at an even (odd) order s, where s is a positive integer. In addition, we consider a more restrictive set

$$\Gamma^{\to s} := \left\{ f \in \mathscr{C}^s \,\middle|\, f \in \Gamma^\ell, \quad \forall \ell = 1, \dots, s \right\},$$

which is the set of all s-time differentiable functions for which their first s successive derivatives alternate in sign. Formally, $\Gamma^{\to s} = \Gamma^1 \cap \Gamma^2 \cap \cdots \cap \Gamma^s$. Hence, we can state the following Theorem and its Corollary.

Theorem 2.1 *For any positive integer s, the two following statements are equivalent:*
(i) *The reduced-form social welfare function in* (1) *satisfies the s-order transfer principle.*
(ii) $W \in \Gamma^s$.

Corollary 2.1 *For any positive integer s, the two following statements are equivalent:*
(i) *The reduced-form social welfare function in* (1) *satisfies all transfer principles up to the sth order.*
(ii) $W \in \Gamma^{\to s}$.

Proof.
These results are due to Fishburn and Willing (1984). ■

We now illustrate the fact that, especially at high orders, the sign of the sth derivative of W is not related to the signs of those of U and V in a straightforward manner. For $s = 1$, where

$$W^{(1)} = V^{(1)} U^{(1)}, \tag{7}$$

the relationship between the signs of $U^{(1)}$ and $V^{(1)}$, and the sign of $W^{(1)}$ is obvious. At higher orders things become more complicated since $W^{(s)} \neq U^{(s)} V^{(s)}$. For $s = 2$, where

$$W^{(2)} = V^{(1)} U^{(2)} + V^{(2)} \left[U^{(1)}\right]^2, \tag{8}$$

it is clear that $U \in \Gamma^2$ and $V \in \Gamma^2$ are not sufficient for $W \in \Gamma^2$. However, $U \in \Gamma^{\to 2}$ and $V \in \Gamma^{\to 2}$ imply $W \in \Gamma^2$, but the reverse is false. For $s = 3$, where

$$W^{(3)} = V^{(1)} U^{(3)} + 3V^{(2)} U^{(1)} U^{(2)} + V^{(3)} \left[U^{(1)}\right]^3, \tag{9}$$

we have that $U \in \Gamma^3$ and $V \in \Gamma^3$ are not sufficient for $W \in \Gamma^3$ (even if one also assumes $U \in \Gamma^1$ and $V \in \Gamma^1$). As mentioned before, sufficiency is obtained if $U \in \Gamma^{\to 3}$ and $V \in \Gamma^{\to 3}$, but the reverse is false. For $s = 4$, where

$$W^{(4)} = V^{(1)} U^{(4)} + V^{(2)} \left[4U^{(1)} U^{(3)} + 3\left[U^{(2)}\right]^2\right] + 6V^{(3)} \left[U^{(1)}\right]^2 U^{(2)} + V^{(4)} \left[U^{(1)}\right]^4, \tag{10}$$

things become clearly even more complicated. Here, assuming that $U \in \Gamma^4$ and $V \in \Gamma^4$ is even less likely to be sufficient for $W \in \Gamma^4$.

The above illustration leads us to conclude that there is no trivial relationship between the sign of $W^{(s)}$ on the one hand, and the signs of $U^{(s)}$ and $V^{(s)}$ on the other hand. Clearly, $U \in \Gamma^s$ and $V \in \Gamma^s$ are together neither sufficient nor necessary for $W \in \Gamma^s$. However, at least for $s = 1, 2, 3, 4$, it is apparent from relations (7) to (10), that $U \in \Gamma^{\to s}$ and $V \in \Gamma^{\to s}$ are together sufficient for $W \in \Gamma^s$ and $W \in \Gamma^{\to s}$ as well. We thus provide the following result, which seems to be reasonably well known, although we could not explicitly find it anywhere in the literature.

Lemma 2.1 *The following statement is true for any positive integer s:*

$\mathbf{H}^s$: $U \in \Gamma^{\to s}$ *and* $V \in \Gamma^{\to s}$ *together imply* $W \in \Gamma^{\to s}$.

Proof.

[$\Rightarrow$] We proceed by mathematical induction, i.e., we first prove that the statement $\mathbf{H}^s$ is true for $s = 1$ and then we prove that if $\mathbf{H}^s$ is assumed to be true for any positive integer s, then so is $\mathbf{H}^{s+1}$. It is apparent from (7) that $\mathbf{H}^1$ is true. Let us now assume that the statement $\mathbf{H}^s$ is true. Remarking that $W^{(s+1)} = \left[V^{(1)}U^{(1)}\right]^{(s)}$ and remembering Leibniz' rule for the sth derivative of the product of two functions we get

$$W^{(s+1)} = \sum_{k=0}^{s} \left(\begin{array}{c} s \\ k \end{array} \right) \left[V^{(1)}\right]^{(s-k)} \left[U^{(1)}\right]^{(k)}.$$

Then, multiplying both sides by $[-1]^{s+1}$ and rearranging terms yields

$$[-1]^{s+1} W^{(s+1)} = -\sum_{k=0}^{s} \left(\begin{array}{c} s \\ k \end{array} \right) [-1]^{s-k+1} V^{(s-k+1)} [-1]^{k+1} U^{(k+1)}.$$

Clearly, the binomial coefficient is positive for all $k = 0, \ldots, s$. If $U \in \Gamma^{\to s+1}$ and $V \in \Gamma^{\to s+1}$, then, for all $k = 0, \ldots, s$, $[-1]^{k+1} U^{(k+1)} \leq 0$ and $[-1]^{s-k+1} V^{(s-k+1)} \leq 0$, implying $W \in \Gamma^{s+1}$. Up to this point we have only demonstrated that $U \in \Gamma^{\to s+1}$ and $V \in \Gamma^{\to s+1}$ together imply $W \in \Gamma^{s+1}$. Finally, as $\mathbf{H}^s$ is supposed to be true, if $U \in \Gamma^{\to s+1}$ and $V \in \Gamma^{\to s+1}$, then, $W \in \Gamma^{\to s}$. This concludes the proof.

[$\not\Leftarrow$] It is apparent from relations (8) to (10). ■

3 Summary and implications

The starting point of this note was the recognition that relating transfer principles to the signs of the successive derivatives of a reduced-form social welfare function is potentially misleading. This is so essentially because any additively separable and symmetric reduced-form social welfare function, such as the one in (1), is a composite function which merges two elements of very different nature: individuals' self-interested preferences over income and, on the other hand, the society's preferences over utility levels. Because assumptions about individuals' preferences – characterized by the shape of U – should be expected to follow from empirical observations, whereas the ones about the society's preferences – characterized by the shape of V – should be expected to be grounded in a theory of distributive justice, it has been argued that it is important to keep the distinction between U and V clear. Hence, recognizing that there was no trivial relationship between the shape of both U and V on the one hand, and the shape of their composition W on the other hand, we have established a simple sufficient condition which ensures a relevant reduced-form social welfare function respecting the generalized transfers principle to be captured.

Our result implies that the answer to the question of whether or not the standard approach – consisting in directly relating transfer principles to the successive derivatives of a reduced-form social welfare function – is problematic typically depends on whether or not the assumption $U \in \Gamma^{\to\infty}$ is empirically relevant. In this respect, it is worth mentioning that in both theoretical and applied works, commonly used Bernoulli utility functions are completely monotone and belong to $\Gamma^{\to\infty}$[4]. According to Lemma 2.1, assumptions about the shape of V then have direct implications regarding transfer principles. To illustrate, if the social welfare function respects the Pareto Principle, i.e. $V \in \Gamma^1$, its reduced form respects the Pen's parade principle, i.e. $W \in \Gamma^1$. In the same way, if the social welfare function is concave, i.e. $V \in \Gamma^2$, its reduced form respects the Pigou-Dalton transfer principle, i.e. $W \in \Gamma^2$, and so on.

However, whenever $U \notin \Gamma^{\to\infty}$, transfer principles may appear to conflict with standard normative assumptions about the shape of the social welfare function. Suppose, for example, that some individuals, may be those with a high level of income, desire less income. Then, any reduced-form social welfare function such as the one in (1) will satisfy the Pen's parade principle only if the Pareto principle is violated. Moreover, in the more realistic case where all individuals desire more income, but where some are risk lovers, even if the social welfare function is concave, its reduced form may fail to satisfy the Pigou-Dalton transfer principle. To illustrate further, if one considers two concave social welfare functions with only one

[4]This property defines the class of utility functions exhibiting "*mixed risk aversion*" (see Brockett and Golden, 1987; Caballé and Pomansky, 1996; Eeckhoudt and Schlesinger, 2006). In particular, all Bernoulli utility functions exhibiting both harmonic and decreasing absolute risk aversion, i.e. *HARA* combined with *DARA*, share this property.

of these supposed to respect the Pareto principle, then if individuals are risk lovers, it is apparent from (8) that the social welfare function which does not respect the Pareto principle will be more likely to generate a reduced-form social welfare function satisfying the Pigou-Dalton transfer principle.

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