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Overlapping coalitions, bargaining and networks

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Abstract
We introduce the game in cover function form, which is a bargaining game of sequential offers for endogenous overlapping coalitions. This extension of games in partition function form removes the restriction to disjoint coalitions. We discuss the existence of equilibria, and we develop an algorithm to compute equilibrium outcomes, under some conditions. We define the key properties that overlapping coalition structures must verify to uniquely identify networks. We show that each network is defined as an equilibrium outcome of a game in cover function form. Our results bridge the two strands of literature devoted to the formation networks and coalitions.
Keywords Overlapping coalitions; Bargaining; Network formation; Coalition formation; Game in cover function form; Symmetric game
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1 Introduction

Agents, such as individuals, countries, or groups, often form coalitions or networks to coordinate their actions.

Coalitions are alliances among agents which differ in goals (Gamson, 1961). They are widespread, for example in politics, environmental issues, provision of public goods, international trade, and have received a huge attention in the economics literature. In economics theory, the formation of coalitions is studied following two distinct approaches: the blocking approach more focused on cooperative games and the bargaining approach, more focused on noncooperative games. However, whatever the approach, the collection of coalitions is mostly a partition of the set of agents. More precisely, all these coalitions are disjoint sets. Therefore, these models fail to address the case of overlapping coalitions. However, a closer look to free trade agreements, environmental agreements, informal insurance groups in developing countries, and political coalitions, reveals that such coalitions overlap.

First, according to the World Trade Organization, more than half of the world trade is now conducted under regional free trade agreements. Since many countries sign more than one of these agreements, regional free trade agreements are overlapping coalitions of countries. Also, environmental agreements involve overlapping coalitions of countries, and such agreements are now widespread due to climate change. Furthermore, in village economies (in developing countries), the absence of formal credit institutions foster the development of informal banking institutions, known as Rotating Saving and Credit Associations.[1] These institutions are mostly overlapping, because several people belong to multiple of them, as reported by van den Brink and Chavas (1997). Finally, Bandyopadhyay and Chatterjee (2006) have reported that in the 2005 state elections in Jharkhand (India), the total number of legislators claimed by the various political groups was greater than the total number of seats in the house. Undoubtedly, some of these political groups overlap. These examples are of invaluable importance, and indicate that the economics theory should pay more attention to overlapping coalitions, as also suggested by Ray (2007).

Of course, overlapping coalitions are studied in computer science and robotics (Kraus et al., 1998; Dang, 2006; Hu et al., 2007), but the problems addressed there are different from those of economic interest. In the economics theory, as far as we know, Boehm (1973) is the first[2] to study the endogenous formation of overlapping coalitions structures. However, he used the term "firm structure" in the light of his general equilibrium model. Next, Myerson (1980) used the term "conference structure" while investigating allocation rules. Recently, following the blocking approach, Albizuri et al. (2006), and Chalkiadakis (2008, 2010) investigate the endogenous formation of overlapping coalitions. However, at the best of our knowledge, nothing similar have been done so far in the bargaining approach. This paper intends to fill this gap.

Networks are made of bilateral interactions (links) between agents. Several different examples fit this general description of networks: friendship, romantic connections, job search, advice seeking, criminal groups, and acquaintanceship. Networks are widely studied in social sciences and have received recently considerable attention in economics. Pioneered by the seminal works of Jackson and Wolinsky (1996), and Bala and Goyal (2000), networks are present in information about job opportunities (Calvo-Armengol and Jackson, 2004), trade of goods in non-centralized markets (Wang and Watts, 2006), provision of mutual insurance in developing countries (Fafchamps and Lund, 2003), research and development collusive alliance among corporations (De Weerdt, 2002), and international

[1] The ROSCA (Rotating Saving and Credit Associations) is known as tontine in Francophone West Africa, Dashi among the Nupe in Nigeria, Isusu among the Ibo and Yoruba, Susu in Ghana, and Ekub in Ethiopia. In Tanzania, it is called Upatu, and it is known as Chilemba in many other parts of East Africa. In other parts of the world, the ROSCA is called Arisan in Indonesia, Pia Huey in Thailand, Ko in Japan, Ho in Vietnam, Kye in Korea, and Hui in Central China. See van den Brink and Chavas (1997) for more details.

[2] I am grateful to an anonymous referee who brings this information to my knowledge .

alliances trading agreements (Furusawa and Konishi, 2007).[3] Though they are different ways of cooperation, networks and coalitions are related. From each network, it is possible to derive a single coalition structure by distinguishing the maximal sets of linked agents known as components. Each component is a coalition, and their collection is a partition of the set of agents. One limitation however, is that the same partition is obtained for a large number of different networks (Bloch and Dutta, 2011), and this prevents the theory of network formation to fully take advantage of the multitude of theoretical tools already developed in the theory of coalition formation. Consequently, a refinement of components that can uniquely identify a network, is highly needed in economics theory. This paper intends also to fill this gap.

The first purpose of this paper is to investigate how overlapping coalitions come to be. We proceed, following the bargaining approach, by developing a new class of bargaining games, namely, games in cover function form. This is an extension of games in partition function form, to the more general setting of overlapping coalitions. Our second purpose is to find a refinement of components that can uniquely represent networks, and define a one-to-one relation between networks and coalition structures.

Our investigations are sanctioned by three main results. The first finding is the extension to our setting, of two principal results obtained in the partition function framework (these are the Theorem 2.1 and 3.1 of Ray and Vohra, 1999). More precisely, our Theorem 1 discusses conditions for the existence of equilibria to games in cover function form. Next, we develop an algorithm, and our Theorem 2 shows that this algorithm computes an equilibrium outcome, subject to its existence. The second finding is our Theorem 3 which identifies the properties overlapping coalition structures must verify, to uniquely represent networks. The strength of this result is to bridge the two strands of the literature on action coordination, which study the formation of networks and coalitions. Finally, the third finding is our Theorem 4 which shows that each network can be identified as an equilibrium outcome of a game in cover function form. This result develops a new rationale to the formation of networks as a sequential process of coalition formation.

This paper is a contribution to the literature of the sequential formation of coalitions. Rubinstein (1982) is the pioneer in the use of an extensive-form bargaining to model a bilateral bargaining process. Next, Chatterjee et al. (1993), and Okada (1996), provide a generalization to a multilateral bargaining (their works differ in the way the proposer is selected in the bargaining process). Following, Bloch (1996), and Ray and Vohra (1999) take a step forward by considering externalities across coalitions (the difference in their works resides in the division of the worth of a coalition). More precisely, their multilateral bargaining takes into account the fact that, the actions of other

[3] These papers are some examples in economics theory where networks are studied. For a complete survey on network formation, see Jackson (2003).

coalitions can affect the worth of already formed ones. In this progression, our paper takes a step forward, by accounting for overlapping coalitions in the multilateral bargaining with externalities.

The remainder of the paper is organized as follows. We build the model in Section 2 by proposing a sequential game that allows the formation of overlapping coalitions. We discuss equilibrium in the sense of existence and computation in Section 3. In Section 4, we discuss the link between coalitions and networks, and we conclude the paper in Section 5. In the appendix, we relegate the proofs of the extensions of the two theorems we borrow from games in partition function form.

2 The model

We devote this section to the definition of a new class of noncooperative bargaining game. The goal is to model the endogenous formation of overlapping coalitions. We begin by defining the value function, and then we define the game in details.

2.1 Cover functions

Let N denote the set of $n \geq 2$ players. We use the term *coalition* to denote any non-empty subset of N. Two distinct coalitions are said to be *overlapping* if they have at least one element in common. A *cover* of N is a collection of coalitions, $\{S_1, S_2, \ldots, S_m\}$, such that $\bigcup_{k=1}^{m} S_k = N$. We denote a typical cover by γ and the set of all covers by Γ. Notice that a partition is a special case of cover, with the restriction that all coalitions are disjoint. Since the term *coalition structure* is used in game theory for partitions, we similarly use *overlapping coalition structure* for covers. We specially do so, whenever we emphasize the structure of a cover, made of multiple possibly overlapping coalitions.

An *embedded coalition*[4] is a pair (S, γ) such that S is a coalition, γ a cover, and $S \in \gamma$. We denote by Σ the set of all embedded coalitions.

Definition 1 A cover function, v, is a nonnegative mapping from Σ into $\Re$.

Notice that a cover function is a value function defined not on coalitions solely, but on embedded coalitions. We do so in the same logic as partition functions, to take into account externalities that can arise during the formation of coalitions. In fact, the formation of a new coalition may affect the "worth" of existing ones. Moreover, if we restrict Γ to the set of all partitions, then a cover function, v, is a partition function. In addition to that, if v is such that for all partition γ and all embedded coalition (S, γ), $v(S, \gamma)$ is independent of γ, then v is a characteristic function. Thus,

[4] We borrow the term embedded coalition from Macho-Stadler et al. (2007).

partition functions and characteristic functions are special cases of cover functions. Owing to that, the game we define (in the next section) on cover functions, encompasses games in characteristic function form and games in partition function form.

2.2 Games in cover function form

To fix ideas, we propose to walk through the following hypothetical situation in a small department of economics. A new research chair in microeconomic theory is created. The only three microeconomic theorists in the department are professors x, y and z. They have to negotiate the formation of research groups and share the research fund provided by the chair. Professor x is a full professor, specialized in cooperative games; y is an aggregate professor, specialized in general game theory; and z is an assistant professor, specialized in noncooperative game theory. The three professors have a meeting, with the protocol that first to take the floor is x, the second y, and the third z. Professor x proposes to y to form a research group $S = \{x, y\}$, and he accepts. Next, professor y proposes to z to form a second research group $S' = \{y, z\}$, and he accepts. When professor z is called to make a proposal, he says that he has nothing new to propose. In other words, z proposes also S' to y and y accepts. At the end of the meeting, two overlapping research groups, S and S' have formed, and the research fund is shared between them. This simple but realistic example depicts a bargaining process that leads to the formation of an overlapping coalition structure. We have this example in mind when we define formally the bargaining game in cover function form.

More generally, the game consists of coalitional bargaining with irreversible agreements. The approach we follow to model negotiation is based on Rubinstein (1982), Chatterjee et al. (1993) and more closely on Ray and Vohra (1999). There is a sequential process made of proposals and responses to proposals. At each stage, a protocol selects a player who proposes to others the formation of a coalition and a sharing out of the worth of this coalition. Players to whom a proposal is made have to respond. They can either accept or reject the proposal. If all of them accept the proposal, then the coalition forms and is irreversible. There is no renegotiation later in the process. The protocol selects another player who makes a new proposal and the game continues the same way. But, if a player rejects the ongoing proposal, he has to make a new proposal. However, rejection is costly for all the players in the game. At the end of the game, if the game ends, a repartition of the players into coalitions obtains, and the worth of each coalition is defined and shared.

Formally, a *game in cover function form* is given by a triple (N, v, ρ), where N is the set of players, v is a cover function, and ρ is a *protocol*. The protocol consists of an *initial* proposer for each coalition and an ordering of *respondents*. For each coalition S, the initial proposer, $\rho^p(S)$, is selected in S to make a proposal, which is to form together with other players, a coalition $S' \subseteq N$. The ordering, $\rho^r(S')$, of the remaining players in S' to whom a proposal is made, defines the order in which they will respond to the proposal. Notice that $\rho^r(S')$ is simply a permutation of the set $S' \setminus \rho^p(S)$. Formally, the protocol is given by the set $\rho \equiv \{(\rho^p(S), \rho^r(S'))\}_{\emptyset \neq S, S' \subseteq N}$.

At a point in the bargaining process where coalitions $\lambda = \{S_1, \ldots, S_L\}$ have already formed, the coalition structure that will form at the end of the game must be compatible with λ. We denote the set of covers that are compatible with λ by $\mathit{\Gamma}_\lambda \equiv \{\gamma \in \mathit{\Gamma} \mid S_l \in \gamma \text{ for all } l = 1, \ldots, L\}$.

We have stated in the introduction that our model accounts for externalities that formation of a particular coalition might have on other coalitions. Therefore, the worth of a particular coalition depends on the whole structure that obtains at the end of the game (this is always the case in games in partition function form) and is not known independently during the bargaining process. With this precision, we define a proposal more formally.

Definition 2 Suppose that the coalitions $\{S_1, \ldots, S_L\}$ have already formed before a player is selected to make a new proposal. Suppose that he proposes to form a coalition S, and let $\lambda = \{S_1, \ldots, S_L, S\}$. A proposal is a pair (S, y_S), such that $y_S \equiv \left\{y_S(\gamma)\right\}_{\gamma \in \mathit{\Gamma}_\lambda}$, and for all $\gamma \in \mathit{\Gamma}_\lambda$, $y_S(\gamma)$ is a vector of size $|S|$, and $\sum_{i \in S} y_S^i(\gamma) = v(S, \gamma)$.

The definition above states that a player selects a coalition to make a proposal to, and he proposes a set of divisions of coalitional worths. Each of these worths to the coalition is conditioned on compatible covers that may form at the end of the game. For each $\gamma \in \mathit{\Gamma}_\lambda$, $y_S(\gamma) = (y_S^i(\gamma))_{i \in S}$.

At each stage $k = 1, 2, \cdots$ of the game, let P_k be the set of players who have not yet been selected to make a proposal, and let $|P_k|$ denote its size. The timing of the game is as follows. The game starts with $P_1 \equiv N$.

At stage k:

- Step (i) The protocol ρ selects an initial player $i = \rho^p(P_k)$ who makes a proposal (S, y_S).
- Step (ii) The players in $S \setminus \{i\}$ respond sequentially in the ordering $\rho^r(S)$. If all of them accept the proposal, then coalition S forms. The game moves to stage $k+1$ with $P_{k+1} = P_k \setminus \{i\}$ and continues as described in Step (i).
- Step (iii) If the proposal is rejected by a player j, then the game moves to stage $k+1$ with $P_{k+1} \equiv P_k$. The time is discounted by a common discount factor $\delta \in (0, 1)$, to account for the cost of a rejection. If the first rejector, j, belongs to P_k, then the game continues as described in Step (i) but the initial proposer becomes $\rho^p(P_k) = j$. Else, the game continues as described in Step (i).

The game ends if $P_{k+1} = \emptyset$. The *outcome* of the game, γ, is the collection of all the coalitions that have formed. Then, each coalition S allocates its realized worth, $v(S, \gamma)$ to its members according to the proposal they have agreed upon.

Remark 1

1. If the bargaining process continues forever, all players receive zero due to discounting.
2. At the stage k of the game, the protocol selects the initial proposer in P_k and he makes a proposal (S, y_S). We do not impose that $S \subseteq P_k$ at the Step (i) of the timing. More clearly, a proposal can be addressed to any player, including players belonging to previously formed coalitions. Thus, some coalitions may intersect and the outcome of the game is indeed an overlapping coalition structure. Notice that existing bargaining games admit only partitions as outcome.

3. The game in cover function form that we define in this paper encompasses games in partition function form, and therefore also games in characteristic function form. To see this, consider the same game with the following restrictions in the timing : (1) we restrict Γ to the set of all partitions; (2) we add to the Step (i) of the timing, the requirement that $S \subseteq P_k$; (3) at Step (ii), the game moves to stage $k+1$ with $P_{k+1} = P_k \setminus S$; (4) the the requirement "Else, the game continues as described in Step (i)" at the end of Step (iii) in the timing, becomes irrelevant (and cancelled) because j is always in P_k. With these restrictions, we obtain a game in partition function form, and an outcome of this restricted game is a partition.

The payoff to a player i in this game, y^i, is the sum (over all the coalitions he belongs to) of all his parts of coalitional worths y^i_S. This is, $y^i \equiv \sum_{S\in\gamma, i\in S} y^i_S$, where γ is the outcome of the game. We make the following assumption to insure that the game ends.

Assumption A.1

A.1.1 *If a player is indifferent between making non-acceptable proposals such that the game continues forever and making an acceptable proposal such that the game ends, he chooses to make the acceptable proposal.*

A.1.2 *If a player is indifferent between rejecting and accepting a proposal, he accepts the proposal.*

Suppose that the cover function is such that all possible payoffs to a player, given the cover function, is zero. This player can act in an ill-intentioned way so as to reduce all other players payoff without increasing his own payoff. He can do so by making always unacceptable proposals (A.1.1) or by rejecting systematically all proposals addressed to him (A.1.2). We make the Assumption A.1 to avoid these situations. In the remainder of the paper, we assume that the Assumption A.1 holds.

3 Equilibrium overlapping coalition structures

3.1 Ingredients

A strategy of a player consists of a complete plan of action that specifies the choices to be made at each node of decision. We study here stationary strategies (as many papers do). More clearly, a player either makes a proposal (possibly probabilistic) or responds to a proposal (possibly probabilistic), conditional only on the *current state of the game*: the proposals that have already been made, the corresponding responses, the coalitions that have already formed, and the ones that can possibly form. The equilibrium concept we consider is therefore, a *subgame stationary perfect equilibrium.* This is a collection of stationary strategies such that there is no history at which a player benefits by deviating from her prescribed strategy.

Before moving to the existence of equilibria it is worthwhile to walk through the following example. For simplicity of the notation in the remainder of the paper, we denote a coalition $S \equiv \{i, j, k, ...\}$ by $S = ijk....$

Example 1
Let's go back to the hypothetical situation we have introduced at the beginning of the Section 2.2. How can we rationalize the choices that the professors x, y and z have made in the formation of research groups? To answer this question, we build a 3-person bargaining game in cover function form that we summarize in the Table 1 below. We bear in mind that professor y is specialized in general microeconomic theory, and also that x and z work in two opposite strands of literature in microeconomics theory.

Table 1: Example of a 3-person game in cover function form

Protocol	Covers	Cover function
$\rho^p(N) = x$	$\gamma_1 = \{N\}$	$v(N, \gamma_1) = 4$
$\rho^p(xy) = x$	$\gamma_2 = \{x, y, z\}$	$v(i, \gamma_2) = 1$ for $i = x, y, z$
$\rho^p(xz) = x$	$\gamma_3 = \{xy, yz\}$	$v(xy, \gamma_3) = v(yz, \gamma_3) = 3$
$\rho^p(yz) = y$	$\gamma_4^k = \{ij, k\}$	$v(ij, \gamma_4^k) = v(k, \gamma_4^k) = 1$ (*)
$\rho^r(N) = y \to z$ (**)		$v(S, \gamma) = 0$ for any other $(S, \gamma) \in \Sigma$

$N = \{x, y, z\}$; (*) i, j, k are such that $\{i, j, k\} = N$; (**) Player y will respond first, followed by player z.

Suppose for simplicity that the coalitional worth is equally shared among players in the coalition.[5] One possible equilibrium action sequence is as follows:

(i) Player x makes the proposal (xy, y_{xy}) to player y , with $y_{xy} = (y_{xy}(\gamma))_{\gamma\in\Gamma_{xy}}$, where
$y_{xy}(\{xy, z\}) = (1/2, 1/2)$
$y_{xy}(\{xy, yz\}) = (3/2, 3/2)$
$y_{xy}(\{xy, \lambda\}) = (0, 0)$ if $\{xy, \lambda\} \in \Gamma_{xy}$.

(ii) Player y accepts the proposal and then the coalition xy forms.

(iii) Player y makes a proposal (yz, y_{yz}) to player z , where
$y_{yz}(\{xy, yz\}) = (3/2, 3/2)$
$y_{yz}(\{xy, yz, \lambda\}) = (0, 0)$ if $\{xy, yz, \lambda\} \in \Gamma_{\{xy.yz\}}$.

(iv) Player z accepts the proposal and the coalition yz forms. Finally, z makes the same proposal as y (it is equivalent to say that player z does not have a new proposal) which he accepts, and the game ends.

It turns out that the cover $\gamma_3 = \{xy, yz\}$ forms in equilibrium. This equilibrium structure of coalitions is indeed an overlapping coalition structure. So, we can trust games in cover function form to rationalize observations similar to the hypothetical example at the beginning of the Section 2.2.

It is worth noting that this specific structure of coalitions from the Example 1 cannot be obtained in the setting of partition functions. In fact, in that setting, we have $v(xy, \gamma_3) = v(yz, \gamma_3) = 0$, because overlapping coalitions are not feasible and because of the nonnegativity of the value function. Therefore, the equilibrium prediction in the partition setting is the grand coalition $\gamma_1 = \{x, y, z\}$. This is, one research group with the individual payoff to a researcher identically equal to $\frac{4}{3}$. There are two problems with this equilibrium prediction. First, the setting of partition function fail to predict the formation of overlapping coalition structures, and therefore are inadequate to rationalize the simple hypothetical observation that we introduced in Section 2.2. Second, with the formation of γ_3 at equilibrium, given all other equilibrium individual payoff to researchers, one can easily construct a division of our equilibrium coalitional worths, such that all the players are better off. Thus, at least in the example in hand, *allowing the formation of overlapping coalitions can lead to Pareto improvements*. For the time being we are not able to show that this result is general, and we are still working on that. This being said, the example shows that, not only the cover function framework responds to a descriptive concern, but also that is seems to be appealing on a normative perspective.

[5] This assumption is made to simplify the computation of the equilibrium in the example. In the paper, the sharing of the coalitional worth is endogenous. Notice however that, in real life people chose equal division as an obvious compromise (Binmore et al., 1985). Finally, equal division is obtained if δ tends to the unity.

3.2 Existence of equilibria

Equilibria in pure strategy do not always exist. This is already shown in the framework of partition functions. Since our setting encompasses the one of partition functions, equilibrium may not exist as well in pure strategies for games in cover function form.
Alike for partition functions, each player in our setting can play a "mixed strategies" in three situations: first in the choice of a coalition to propose, second in the distribution of the coalitional worth once a coalition is chosen, and finally in a choice of the response to give to a proposal. The result that we have in the Theorem 1 below, is the existence of equilibrium, not in completely mixed strategies, but with a mild degree of mixing. The same theorem is obtained by Ray and Vohra (1999) for partition functions.

Theorem 1 *Theorem 2.1 by Ray and Vohra (1999) remains true for games in cover function form. This is, there exists a stationary subgame perfect equilibrium where the only source of mixing is in the choice of a coalition by each proposer.*

We construct the proof along the lines of Ray and Vohra (1999). Notice however that three additional complexities arise in the case of cover functions. First, players who already form coalitions remain in the game and may make proposal or receive proposals. Second, a player may reject a proposal but may not necessary be the next proposer. Third, the payoff to players who may form multiple coalitions depends not on a proposal in hand, but on the aggregate payoffs from each coalition. Therefore, the strategic object of decision is completely different. All these complexities make the proof more hard. But, since the steps are more or less the same as in the framework of partitions, we relegate the details of the proof to the Appendix section.

3.3 Symmetric cover functions

The analysis of equilibria for general games in cover function form is quite complex. Our objective in this section is to develop a procedure for symmetric games, to compute equilibrium outcomes.
Generally speaking, a game is symmetric if the value function does not depend on the individual identity (the name) of the players. However, accounting for that in a setting where some players may belong to more than one coalitions is far from being obvious. To begin with, we recall the definition of symmetry in the setting of partition functions.

Symmetric partition function (Ray and Vohra, 1999)
Let N be a set of n players, and π a partition of N. If $\pi = \{S_1, S_2, \ldots, S_m\}$ with $|S_i| = s_i$, then the

numerical coalition structure of π is $\mathbf{n}(\pi) \equiv (s_1, s_2, \ldots, s_m)$. A *partition function* v is *symmetric* if for a given partition π and a coalition $S_i \in \pi$, $v(S_i, \pi) = v(s_i, \mathbf{n}(\pi))$.

Thus, the numerical coalition structure represents the equivalence class of coalition structures relative to the symmetric partition function, and the list of the size of the coalitions is sufficient to define outcomes. We need to go beyond that in our setting. Notice in the definition above, that the numerical structure, $\mathbf{n}(\pi)$, implies that $\sum_{i=1}^{m} s_i = n$. But for a cover function $\gamma \equiv \{S_1, \ldots, S_m\} \in \Gamma$, if $\mathbf{n}(\gamma) \equiv (s_1, ..., s_m)$ is the associated numerical coalition structure, then $\sum_{i=1}^{m} s_i \geq n$. This inequality is strict at least two coalitions overlap. Thus, we cannot use numerical coalition structures alone to define symmetric cover functions. Instead, we need to construct an additional tool that will take into account the fact that coalitions overlap. In the sequel, we set up the ingredients that will allow us to properly define symmetric cover functions.

For each coalition S in a cover γ, we identify players in S that belong to more than one coalitions. We name them *overlapping players*. Then we count the number of these players who belong to a given number of distinct coalitions. Doing so, we define couples, (o_i, c_i), of integers to be interpreted as o_i players are members of c_i distinct coalitions. We name the collection of all such couples of integers, $\mathbf{o}(S) \equiv \big\{(o_1, c_1), \ldots, (o_l, c_l)\big\}$, by the *overlapping status* of the coalition S. We do the same exercise for all distinct coalitions in γ, and we obtain a unique structure for each embedded coalition.

Definition 3 Let $\gamma \equiv \{S_1, \ldots, S_m\}$ be a cover and $s_i = |S_i|$ the size of the coalition S_i, $i = 1, 2, \ldots, m$. The representative overlapping coalition structure of γ is defined by

$$\mathbf{r}(\gamma) \equiv \Big\{\{s_i; \mathbf{o}(S_i)\}, i = 1, 2, \ldots, m\Big\}$$

Remark 2

1. By definition, covers with the same numerical structure such that for each coalition of the same size, the overlapping status is also the same, admit the same representative overlapping coalition structure.
2. If γ is a partition, then for all $S_i \in \gamma$, $\mathbf{o}(S_i) = \big\{(0; 0)\big\}$. In this case the representative coalition structure is equivalent to the numerical coalition structure.

All the ingredients are now in place for the definition of symmetric cover function. Nevertheless, it is worthy to walk through the example below, to get familiar with our notations.

Example 2 $N = \{x, y, z, a, b\}$. In the Table 2 below, we write down the representative overlapping coalition structures for five distinct covers of N, and we compare with the numerical coalition structures (defined for partition functions).

For γ_1, the two notions are equivalent because there are no overlapping players. Notice that γ_3

Table 2: representative overlapping coalition structure

γ	$\mathbf{r}(\gamma)$	$\mathbf{n}(\gamma)$
$\gamma_1 = \{xyzab\}$	$\Big\{\{5;\{(0,0)\}\}\Big\}$	(5)
$\gamma_2 = \{xyz, zab\}$	$\Big\{\{3;\{(1,2)\}\};\{3;\{(1,2)\}\}\Big\}$	$(3,3)$
$\gamma_3 = \{xyz, xya, xyb\}$	$\Big\{\{3;\{(2,3)\}\};\{3;\{(2,3)\}\};\{3;\{(2,3)\}\}\Big\}$	$(3,3,3)$
$\gamma_4 = \{xzb, yzb, yab\}$	$\Big\{\{3;\{(1,2),(1,3)\}\};\{3;\{(1,3),(2,2)\}\};\{3;\{(1,2),(1,3)\}\}\Big\}$	$(3,3,3)$
$\gamma_5 = \{xya, xyz, xzb\}$	$\Big\{\{3;\{(1,2),(1,3)\}\};\{3;\{(1,3),(2,2)\}\};\{3;\{(1,2),(1,3)\}\}\Big\}$	$(3,3,3)$

and γ_4 have the same numerical coalition structure but distinct representative overlapping coalition structures. For example, in γ_3, coalition xyz is of size 3, and contains 2 overlapping players who are in 3 distinct coalitions, the same for xya and xyb. In γ_4, coalition xzb is also of size 3, but contains 1 overlapping layer who is in 2 distinct coalitions and a second one in 3 distinct coalitions.

The Definition 4 below identifies symmetric cover functions as value functions such that, given a cover, the worth of a coalition depends *solely* on its representative overlapping coalition structures.

Definition 4 A cover function v is symmetric if for each embedded coalitions (S,γ),

$$v(S_i,\gamma) = v\Big(\{s_i;\mathbf{o}(S_i)\},\mathbf{r}(\gamma)\Big).$$

Notice that, given a cover, two coalitions with the same number of players may not necessary have the same worth. The distinction is due to the overlapping status. Obviously this problems is irrelevant in the setting of partitions. Suppose we have used the numerical coalition structure to define symmetric cover functions. Since γ_3, γ_4 and γ_5, defined in the Table 2 above, have the same numerical coalition structure, then for a symmetric cover function v, all the $v(S,\gamma)$ must be identical for all $\gamma \in \{\gamma_3,\gamma_4,\gamma_5\}$. This is not the case in our setting. In fact, for a symmetric cover function v, we have $v(xyz,\gamma_3) = v(xya,\gamma_3) = v(xyb,\gamma_3)$, because all these coalitions have the same representative overlapping coalition structure. But, given the cover γ_4, coalitions xzb and yab must have the same worth and this value may be different from the worth of the coalition yzb. Furthermore, since γ_4 and γ_5 have the same representative overlapping coalition structure, then the value of coalitions xzb and yab given γ_4 or γ_5 must be the same.

Finally, we define a *symmetric game in cover function form* as a game in cover function form, where the cover function is symmetric.

3.4 Computation of equilibrium overlapping coalition structures

In the following, we extend the algorithm proposed by Ray and Vohra (1999) for symmetric partition functions to our setting. We use, for simplicity of notation, some expressions without their arguments.

Consider a symmetric game in cover function form, $\{N, v, \rho\}$, with n players. For each representative overlapping coalition structure $\mathbf{r}$ at the stage $k_{\mathbf{r}}$ of the game such that $P_{k_{\mathbf{r}}} \geq 0$, the algorithm consists of the construction of two sets:

1. A representative overlapping coalition structure, $\{s; o\}_{\mathbf{r}}$, where s and $\mathbf{o}$ are respectively the size and the overlapping status of a coalition S. This coalition represents the next equilibrium coalition to form in the bargaining process.
2. A set $c_{\mathbf{r}}$ that completes $\mathbf{r}$ to obtain a representative equilibrium overlapping coalition structure for all the game.

To finish, we compute $\{s; \mathbf{o}\}$ for the empty set and we continue repeatedly to generate a particular representative overlapping coalition structure, $\mathbf{r}^*$, for our game. We show later in the paper that under some conditions, $\mathbf{r}^*$ is an equilibrium representative overlapping coalition structure for the game $\{N, v, \rho\}$.

Step 1 For $\mathbf{r}$ such that the size of $P_{k_{\mathbf{r}}} = 1$, define $\{s; \mathbf{o}\}_r$ as the argument of the maximum of $\frac{v\left(\{s;\mathbf{o}\},c_{\mathbf{r}}\right)}{s}$ where $c_{\mathbf{r}}$ is the representative overlapping coalition structure obtained by completing $\mathbf{r}$ with $\{s; \mathbf{o}\}$. We denote this operation by $c_{\mathbf{r}} \equiv \mathbf{r} \bullet \{s; \mathbf{o}\}_{\mathbf{r}}$. [6] If there exists more than one arguments of the maximum, we chose the one with the highest s.[7] The game ends at this step because the last proposer cannot make an unacceptable proposal.[8]

Step 2 Suppose that we have defined $\{s; \mathbf{o}\}_{\mathbf{r}}$ and $c_{\mathbf{r}}$ for each stage $k_{\mathbf{r}}$ of the game such that the size of $P_{k_{\mathbf{r}}} = 1, 2, \cdots, m$, with $m < n$.

Step 3 For each $\mathbf{r}$ such that the size of the set of proposers $P_{k_{\mathbf{r}}} = m + 1$, define $\{s; \mathbf{o}\}_r$ as the argument of the maximum of $\frac{v\left(\{s;\mathbf{o}\},c_{\mathbf{r'}}\right)}{s}$ where $\mathbf{r'} \equiv \mathbf{r} \bullet \{s; \mathbf{o}\}_{\mathbf{r}}$ (notice that $c_{\mathbf{r'}}$ is defined at the **Step 2** of the algorithm). If there exists more than one arguments of the maximum, we chose the one with the highest s.

Step 4 Define a representative overlapping coalition structure for the entire game by $\mathbf{r}^*$, such that $\mathbf{r}^* \equiv c_{\emptyset}$.

[6] Notice that if the new coalition was already proposed, then $\mathbf{r} \bullet \{s; \mathbf{o}\}_{\mathbf{r}} = \mathbf{r}$.

[7] We take the highest integer to have a unique value for $\mathbf{r}$ so that the algorithm generates a unique representative overlapping coalition structure at the end.

[8] In the proof of Theorem 1, we show that due to the discount factor, the last proposer cannot make an unacceptable proposal.

Notice that one important thing in this algorithm is the construction of all candidate sets $c_{\mathbf{r}}$ corresponding to a given $\mathbf{r}$. Notice that the maximal number of coalitions is n (because each coalition follows a proposal) and that a cover cannot contain more than n players. Therefore, we must be able to count the number of distinct players in a collection of coalitions during the bargaining process. This is an easy task for partition functions, because it is sufficient to sum the figures in the numerical coalition structures. For our setting, we need to define a counter for the same purpose. The Lemma 1 above defines a functional K that counts the exact number of distinct players in a representative overlapping coalition structure.

Lemma 1

Consider a collections of coalitions λ and the corresponding representative overlapping coalition structure $\boldsymbol{r}(\lambda)$. Let $\boldsymbol{o}(\lambda) \equiv \{(o_1, c_1), \ldots, (o_l, c_l)\}$ be the overlapping status of λ. The integer $K_{\boldsymbol{r}(\lambda)} \equiv \sum_{i=1}^{m} s_i - \sum_{j=1}^{l}(c_j - 1)o_j$ counts the exact number of distinct players in λ.

The proof of this lemma is straightforward. If we sum all the s_i's in the overlapping status of λ, then there are multiple counts for overlapping players. We deduct from this sum the number of times each overlapping player is counted again and we obtain the exact number of distinct players in λ.

In the Example 3 below, we compute the integer K for some representative overlapping coalition structures.

Example 3 $N = \{x, y, z, a, b\}$. In the Table 3, we use the integer K, defined in the Lemma 1 above, to compute the exact number of distinct players in the five covers we have introduced in the previous example.

Table 3: counting the players given a representative coalition structure

γ	$\mathbf{n}(\gamma)$	$\sum s_i$	$\mathbf{o}(\gamma)$	$\sum(c_j - 1)o_j$	$K_{\mathbf{r}(\gamma)}$
$\gamma_1 = \{xyzab\}$	(5)	5	$\{(0,0)\}$	0	5
$\gamma_2 = \{xyz, zab\}$	$(3,3)$	6	$\{(1,2)\}$	1	5
$\gamma_3 = \{xyz, xya, xyb\}$	$(3,3,3)$	9	$\{(2,3)\}$	4	5
$\gamma_4 = \{xzb, yzb, yab\}$	$(3,3,3)$	9	$\{(1,3),(2,2)\}$	4	5
$\gamma_5 = \{xya, xyz, xzb\}$	$(3,3,3)$	9	$\{(1,3),(2,2)\}$	4	5

Before the formal proof that the algorithm computes indeed an equilibrium representative structure, we provide the following example to help the reader walk through the algorithm .

Example 4 We take a symmetric version of the game that we have introduced in Example 1, $N = \{x, y, z\}$. First, we use the following Table 4, to summarize a symmetric cover function. Thereafter, we use the Table 5 to implement the algorithm.

Table 4: a 3-person symmetric cover function

γ	$\mathbf{r}(\gamma)$	$v(\{s; \mathbf{o}\}, \mathbf{r})$
$\gamma_1 = \{N\}$	$\{3; (0,0)\}$	4
$\gamma_2 = \{x, y, z\}$	$\{\{1; (0,0)\}, \{1; (0,0)\}, \{1; (0,0)\}\}$	$(1, 1, 1)$
$\gamma_3 = \{xy, yz\}$	$\{\{2; (1,2)\}, \{2; (1,2)\}\}$	$(3, 3)$
$\gamma_4^k = \{ij, k\}$ (*)	$\{\{2; (0,0)\}, \{1; (0,0)\}\}$	$(1, 1)$
$\gamma_5 = \{xz, yz\}$	$\{\{2; (1,2)\}, \{2; (1,2)\}\}$	$(3, 3)$
$\gamma_6 = \{xy, xz\}$	$\{\{2; (1,2)\}, \{2; (1,2)\}\}$	$(3, 3)$
		0 for any other $\mathbf{r}$

(*) i, j, k are such that $\{i, j, k\} = N$.

For the same arguments as in Example 1, obviously $r^*(\gamma) \equiv \big\{\{2; (1,2)\}, \{2; (1,2)\}\big\}$ is the equilibrium representative coalition structure. The outcome is one of the covers, in $\{\gamma_3, \gamma_5, \gamma_6\}$. The exact one that will form depends on the protocol.

Now we implement the algorithm to compare its predictions with the equilibrium outcome of the game. We use for this purpose the Table 5 below.

The Table 5 in the Example 4, contains only representative overlapping coalitions such that the value function non null and all of them have five players ($K = 5$). Those of null values are not necessary for this example and will needlessly overload the table.

The first series of rows correspond to all possible representative overlapping coalitions structures, $\mathbf{r}$, at the stage k of the game such that the size of P_k is 1. The algorithm generates $\{s; \mathbf{o}\}_r$ and c_r that we mark with stars in the Table 5. This operation is repeated in the second series of rows, for the size of P_k equal 2 and the third series for the size of P_k equal 3. The prediction of the algorithm is $\mathbf{r}^* \equiv c_\emptyset = \big\{\{2; (1,2)\}, \{2; (1,2)\}\big\}$ and it is the representative overlapping coalition of the equilibrium outcome of the symmetric game $\{N, v, \rho\}$.

Table 5: implementation of the algorithm

$\lvert P_{k_{\mathbf{r}}}\rvert$	$\mathbf{r}$	$c_{\mathbf{r}}$	$\frac{v(\{s;\mathbf{o}\},c_{\mathbf{r}})}{s}$	$\{s;\mathbf{o}\}_{\mathbf{r}}$
1	$\Big\{\{3;\{(0,0)\}\}\Big\}$	$\Big\{\{3;\{(0,0)\}\}\Big\}^*$	$\frac{4}{3}\simeq 1.33$	$\Big\{\{3;\{(0,0)\}\}\Big\}^*$
1	$\Big\{\{2;\{(0,0)\}\};\{1;\{(0,0)\}\}\Big\}$	$\Big\{\{2;\{(0,0)\}\};\{1;\{(0,0)\}\}\Big\}^*$	$\frac{1}{2}=0.5$	$\Big\{\{3;\{(0,0)\}\}\Big\}^*$
1	$\Big\{\{2;\{(1,2)\}\};\{2;\{(1,2)\}\}\Big\}$	$\Big\{\{2;\{(1,2)\}\};\{2;\{(1,2)\}\}\Big\}^*$	$\frac{3}{2}=1.5$	$\{2;\{(1,2)\}\}^*$
1	$\Big\{\{2;\{(0,0)\}\}\Big\}$	$\Big\{\{2;\{(0,0)\}\};\{1;\{(0,0)\}\}\Big\}$	$\frac{1}{1}=1$	
1	$\Big\{\{2;\{(0,0)\}\}\Big\}$	$\Big\{\{2;\{(1,2)\}\};\{2;\{(1,2)\}\}\Big\}^*$	$\frac{3}{2}=1.5$	$\{2;\{(1,2)\}\}^*$
1	$\Big\{\{1;\{(0,0)\}\};\{1;\{(0,0)\}\}\Big\}$	$\Big\{\{1;\{(0,0)\}\};\{1;\{(0,0)\}\};\{1;\{(0,0)\}\}\Big\}^*$	$\frac{1}{1}=1$	$\{1;\{(0,0)\}\}^*$
2	$\Big\{\{3;\{(0,0)\}\}\Big\}$	$\Big\{\{3;\{(0,0)\}\}\Big\}^*$	$\frac{4}{3}\simeq 1.33$	$\{3;\{(0,0)\}\}^*$
2	$\Big\{\{2;\{(0,0)\}\}\Big\}$	$\Big\{\{2;\{(0,0)\}\};\{1;\{(0,0)\}\}\Big\}$	$\frac{1}{1}=1$	
2	$\Big\{\{2;\{(0,0)\}\}\Big\}$	$\Big\{\{2;\{(1,2)\}\};\{2;\{(1,2)\}\}\Big\}^*$	$\frac{3}{2}=1.5$	$\{2;\{(1,2)\}\}^*$
2	$\Big\{\{2;\{(0,0)\}\}\Big\}$	$\Big\{\{2;\{(1,2)\}\};\{2;\{(1,2)\}\}\Big\}$	$\frac{3}{2}=1.5$	$\{2;\{(0,0)\}\}$
2	$\Big\{\{1;\{(0,0)\}\}\Big\}$	$\Big\{\{1;\{(0,0)\}\};\{1;\{(0,0)\}\};\{1;\{(0,0)\}\}\Big\}^*$	$\frac{1}{1}=1$	$\{1;\{(0,0)\}\}^*$
2	$\Big\{\{1;\{(0,0)\}\}\Big\}$	$\Big\{\{2;\{(0,0)\}\};\{1;\{(0,0)\}\}\Big\}$	$\frac{1}{2}=0.5$	
3	$\emptyset$	$\Big\{\{3;\{(0,0)\}\}\Big\}$	$\frac{4}{3}\simeq 1.33$	
3	$\emptyset$	$\Big\{\{2;\{(1,2)\}\};\{2;\{(1,2)\}\}\Big\}^*$	$\frac{3}{2}=1.5$	$\{2;\{(0,0)\}\}^*$
3	$\emptyset$	$\Big\{\{1;\{(0,0)\}\};\{1;\{(0,0)\}\};\{1;\{(0,0)\}\}\Big\}$	$\frac{3}{2}=1.5$	

$$\boxed{\mathbf{r}^* = \{\{2;(1,2)\},\{2;(1,2)\}\}}$$

In the remainder of this section, we show formally that under some conditions (Assumption A.2 below), the prediction of the algorithm is indeed an equilibrium representative overlapping coalition structure.

Assumption A.2

For $\mathbf{r}$ such that $|P_{k_{\mathbf{r}}}| > 1$, there exists a set $\{s;\mathbf{o}\}$ such that $v(\{s;\mathbf{o}\},c_{\mathbf{r'}}) > 0$ for all $c_{\mathbf{r'}}$ where $\mathbf{r'} = \mathbf{r} \bullet \{s;\mathbf{o}\}$.

This positivity result is a technical requirement for the proof of the Theorem 2 below. Notice however that this is a very weak assumption. In fact it is sufficient to consider values of zero as positive and as small as possible to meet this requirement.

Theorem 2 *Under the Assumption A.2, Theorem 3.1 by Ray and Vohra (1999) remains true for symmetric games in cover function form. This is, there exists* $\delta^* \in (0,1)$ *such that for all* $\delta \in (\delta^*, 1)$*, the representative overlapping coalition structure corresponding to any equilibrium in which an acceptable proposal is made with positive probability at any stage, is* $\boldsymbol{r}^*$.

Once again, the steps of the proof are similar to the partition function case. There are big differences however, due to the divergences in the notion of symmetry in the two frameworks. The adaptation from one setting to another is far from being an obvious task. But the spirit of the proofs are identical. For this reason, we relegate once again the details of the proof to the Appendix section.

Now that we introduce the class of games in cover function form, and provide an algorithm to compute equilibrium outcomes for symmetric games, we move in the remainder of the paper to some applications to networks.

4 Networks vs overlapping coalition structures

4.1 Ingredients from the theory of networks

Let N be the set of players. A network g is a list of unordered pairs of players $\{i,j\}$. For simplicity in the remainder of the paper, we write ij instead of $\{i,j\}$. The Figure 1 above shows examples of networks for five players. The nodes are the players and the edges represent links between players.

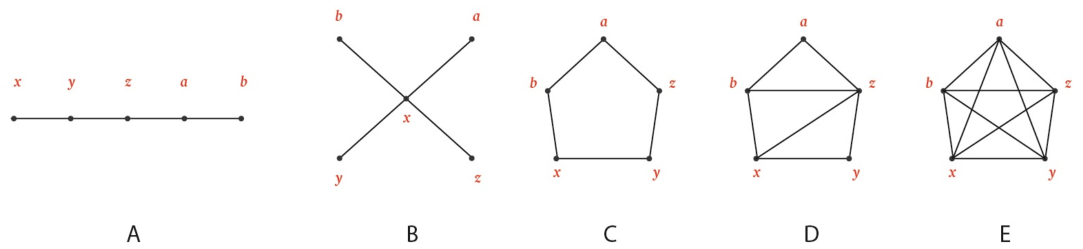

Figure 1: examples of 5-person networks

A *path* of length L between players i and j in the network g is a sequence $i_1, \ldots, i_{L+1}$ such that $i_l i_{l+1} \in g$ for each $l \in \{1, \ldots, L\}$, with $i_1 = i$ and $i_{L+1} = j$.
Given a network g, two distinct players, i and j are *linked* if there exists a path between them. If

there exists a path of length one ($L = 1$) between two linked players (therefore, this pair of players is in g), then they are *directly linked*. Else, two linked players are *indirectly linked*. A *component* is a maximal set of linked players in g.

The notion of component is widely used in the economics literature to identify coalitions formed by linked players. Notice however that, even if all the players in a component are linked, some are directly linked and others indirectly. In this paper, our interest resides only in directly linked players. This implies a refinement of components.

A *clique* is a maximal set of directly linked players in g. We denote by γ_g the collection of all the cliques of g.

For simplicity, we restrict ourselves to networks with only one component. These are networks where all the players are linked. This is not a limitation to our paper because all our results are valid for all networks.

4.2 Networks as overlapping coalition structures and vice versa

We introduce this section by the Example 5 below. For each network in the Figure 1, we construct the corresponding cliques.

Example 5 Let $N = \{x, y, z, a, b\}$ be the set of all the players. We use the Table 6 below to summarize the construction the cliques for each network g in the Figure 1.

Table 6: decomposition of 5-person networks in cliques γ_g

Network g	Cliques S_i	γ_g	Components
A	$S_1 = xy; S_2 = yz; S_3 = za; S_4 = ab$	$\gamma_A = \{xy, yz, za, ab\}$	$\{xyzab\}$
B	$S_1 = xy; S_2 = xz; S_3 = xa; S_4 = xb$	$\gamma_B = \{xy, xz, xa, xb\}$	$\{xyzab\}$
C	$S_1 = xy; S_2 = yz; S_3 = za; S_4 = ab; S_5 = xb$	$\gamma_C = \{xy, yz, za, ab, xb\}$	$\{xyzab\}$
D	$S_1 = xyz; S_2 = xzb; S_3 = zab$	$\gamma_D = \{xyz, xzb, zab\}$	$\{xyzab\}$
E	$S = xyzab$	$\gamma_E = \{xyzab\}$	$\{xyzab\}$

Definition 5 A cover representation for a network g is the collection of all the cliques, namely γ_g.

The following Lemma 2 points out one direction of the link between networks and overlapping coalition structures.

Lemma 2 *For each network g, the cover representation γ_g is unique, this is $g \neq g' \Leftrightarrow \gamma_g \neq \gamma_{g'}$, and is an overlapping coalition structure.*

Proof.

We prove this lemma in three parts.

First, for each network g, each maximal set of directly linked players is unique. Thus, γ_g is trivially unique by definition.

The second part of the proof shows that distinct networks admit distinct cover representations. Consider two networks, g and g' such that $g \neq g'$ and the corresponding cover representations γ_g and $\gamma_{g'}$ respectively. Suppose by absurd that $\gamma_g = \gamma_{g'}$. Because $g \neq g'$, there exists a pair of players, ij such that $ij \in g$ and $ij \notin g'$. Consider a clique S such that $i \in S$ and $j \in S$. Obviously, $S \in \gamma_g$ and therefore $S \in \gamma_{g'}$ because $\gamma_g = \gamma_{g'}$ by assumption. Thus, $ij \in g'$ which is absurd.

The third part of the proof shows that γ_g is an overlapping coalition structure.

Notice first that each player belongs to a least one clique because each players has at least one link in the network g. Thus, $\bigcup_{S\in\gamma_g} S = N$. Furthermore, for each three distinct players $i, j, k \in N$ such that i is directly linked to j and j is directly linked to k but k is indirectly linked to i, ij and jk belong to two distinct cliques. But these cliques overlap because each one contains j . □

Notice that covers show much more complexity in there structure than networks. More precisely, the number of distinct covers is much more higher than the number of networks for the same set of players. The following definition identifies some specific covers, and we show thereafter that they can uniquely define networks.

Definition 6 A cover γ of N is parsimonious if and only if

1. $\gamma = \{N\}$ or
2. $|S| = 2$ for all $S \in \gamma$, and there exists a unique couple $S, S' \in \gamma$ such that $S \cap S' = \{i\}$, for all $i \in N$ or

3.(i) $S \nsubseteq S'$ for all distinct pairs of coalitions $S, S' \in \gamma$, and

3.(ii) there exists an ordering $i_1, i_2, \ldots, i_m$ of the indexes of coalitions in γ such that, $S_{i_2} \setminus S_{i_1} \neq \emptyset$, $S_{i_3} \setminus \{S_{i_2} \cup S_{i_1}\} \neq \emptyset$,..., $S_{i_m} \setminus \{S_{i_{m-1}} \cup S_{i_{m-2}} \cup \ldots \cup S_{i_1}\} \neq \emptyset$, and $\gamma = \{S_{i_1}, S_{i_2}, \ldots, S_{i_m}\}$.

In this definition, requirements 1, 2 and 3 are exclusive. In fact, from 2, there exists n distinct coalitions. But each one is included in the union of the $n-1$ remaining. Therefore requirement 3.(ii) of the definition can not hold. On the other hand, if requirement 3.(ii) holds and each coalition contains two players, with no inclusion, and requirement 2, then each coalition must have at least two players and for this reason we may have at least $n+1$ players, and this is a contradiction. Also, it is obvious that if requirement 1 is verified, there is only one coalition, and therefore 2 and 3 do not hold.

Notice that all the covers in the Example 5 above are parsimonious. First, γ_E follows the requirement 1 of the definition. Second, γ_C follows the requirement 2 of the definition. Third, γ_A, γ_B, and γ_D follow the requirements 3.(i) and 3.(ii). For γ_A, there is no inclusion, and $S_2 \setminus S_1 \neq \emptyset$; $S_3 \setminus S_2 \cup S_1 \neq \emptyset$;$S_4 \setminus S_3 \cup S_2 \cup S_1 \neq \emptyset$. The same result obtains for γ_B. For γ_D, there is no inclusion and $S_2 \setminus S_1 \neq \emptyset$; $S_3 \setminus S_2 \cup S_1 \neq \emptyset$.
Besides, consider the following five-person covers: $\gamma = \{xy, xyz, ab\}$, $\gamma' = \{xyz, yza, zab, xab, xyb\}$. None of them are parsimonious. First, for γ, $xy \subset xyz$ is a violation of requirement 3.(i) , and it also obviously violates requirements 1 and 2. Second, for γ', each coalition is included in the union of the remaining. Therefore, whatever the ordering, $S_{i_5} \setminus \{S_{i_4} \cup S_{i_3} \cup \ldots \cup S_{i_1}\} = \emptyset$. This is a violation of requirement 3.(ii), and it also obviously violates requirement 1 and 2.

The theorem 4 below defines a one-to-one relation between networks and overlapping coalition structures.

Theorem 3 *A cover γ is parsimonious if and only if there exists a unique network g such that $\gamma = \gamma_g$.*

Proof.
We prove the theorem in three parts, each part being for each requirement of the Definition 6 of (parsimonious covers).
Part 1.
If g is the complete network, this is the network where all the players are directly linked, then $\gamma_g = \{N\}$. Also, for the cover $\gamma = \{N\}$, we uniquely associate the complete network.
Part 2.
Let $g \in G$ be a circle. This is the network such that each node has exactly two neighbors. The circle has a single cycle, this is a path $i_1, i_2, ..., i_m, i_{n+1}$, with $i_{n+1} = i_1$. All the cliques of g has the form $S_j = \{i_j, i_{j+1}\}, 1 \leq j \leq n$, and the cover representation of g is $\gamma_g \equiv \{S_1, S_2, ..., S_n\}$. For all $j < n$, $|S_j| = 2$, $S_j \cap S_{j+1} = \{i_j\}$. Furthermore, $|S_n| = 2$, and $S_n \cap S_1 = \{i_1\}$ and these coalitions are the unique ones that intersect in $\{i_1\}$.
Besides, let γ be a cover such that $\gamma \neq \emptyset$. By the uniqueness requirement in 2 of the definition of

parsimonious, we have γ contains n distinct coalitions of size two each. Choose $S_1 \equiv \{i_1, i_2\} \in \gamma$. Set S_2 as the coalition in γ such that $S_2 \equiv \{i_2, i_3\}$ with $i_3 \neq i_1$. Continue step by step, choosing at step k, $S_k \equiv \{i_k, i_{k+1}\}$ such that $i_{k+1} \notin \{i_2, i_3, \dots, i_{k-1}\}$. At step n, $i_{n+1} = i_1$. The path $i_1, i_2, ..., i_n, i_{n+1}$ is a cycle and the induced network is a circle.

Part 3.

Let $g \in G$ be a network of N. Notice that if $n = 3$ and g is a circle, $\gamma_g = \{N\}$ and therefore γ_g is parsimonious.

Now suppose $n > 3$ and consider a clique S_1 of g. Either $S_1 = N$ and γ_g is parsimonious, or there exists $i_1 \in S_1$ and $i_2 \in N \setminus S_1$ such that the link $i_1 i_2 \notin g$. Let S_2 be a clique of g that contains i_2. Either $S_2 \cup S_1 = N$, or there exists $i_2' \in S_2$ and $i_3 \in N \setminus (S_2 \cup S_1)$ such that the link $i_3 i_2' \notin g$. Let S_3 be a clique of g that contains i_3. Step by step, we construct an ordering of the indexes of the coalitions $S_1, S_2, \dots$ that satisfy the parsimonious requirement 3.(ii). As N is finite, there exists a step m such that $N \setminus \bigcup_{1 \leq j \leq m} S_j = \emptyset$. Thus, $\bigcup_{1 \leq j \leq m} S_j = N$ and the if part of the proof is over if $\gamma_g = \{S_1, S_2, \dots, S_m\}$. If not, there exists $S \in \gamma_g$ such that $S \notin \{S_1, S_2, \dots, S_m\}$. In the following we test S against the coalitions backward from S_m to S_1. We say that S_m passes the test if there exists $i \in S$ such that $i \in S_m$ and $i \notin \bigcup_{1 \leq j \leq m-1} S_j$. Thus, $S_m \setminus S \neq \emptyset$, $S \setminus \bigcup_{1 \leq j \leq m-1} S_j \neq \emptyset$, and $S_m \setminus (S \cup \bigcup_{1 \leq j \leq m-1} S_j) \neq \emptyset$. In contrary, if S_m fails the test, then $S \subset \bigcup_{1 \leq j \leq m-1} S_j$, and S_{m-1} takes the test. If S_{m-1} also fails the test, then we proceed with S_{m-2}. If coalitions in γ_g fail the test up to S_2, then S_2 can not fail the test, otherwise $S \subset S_1$ and this is not possible because S and S_1 are cliques. Therefore, there exists $k \in \{2, 3, \dots, m\}$ such that $S_k \setminus S \neq \emptyset$, $S \setminus \bigcup_{1 \leq j \leq k-1} S_j \neq \emptyset$, and $S_k \setminus (S \cup \bigcup_{1 \leq j \leq k-1} S_j) \neq \emptyset$. Thus, rank the indexes of the coalitions as: for $1 \leq j < k-1$, $S_{i_j} \equiv S_j$; $S_{i_k} \equiv S$; and for $k \leq j \leq m$, $S_{i_{j+1}} \equiv S_j$. If $\gamma_g = \{S_{i_1}, S_{i_2}, \dots, S_{i_{m+1}}\}$, then γ_g verifies the requirement 3.(ii) of parsimonious property. If not, we proceed the same way as we have done previously with the new coalition S until we have the ordering of all the coalitions of γ_g that satisfies the requirement 3.(ii) of parsimonious property. And we continue the same way until no S remains (the number of distinct cliques is finite).

Furthermore, since cliques are maximal sets of directly linked players, no inclusion is possible. Therefore, the parsimonious requirement 3.(i) is satisfied and this ends the if part of Step 2 of the proof.

For the only if part, let γ be a parsimonious cover, following requirement 3.(ii). Consider the ordering of the indexes of the coalitions of γ, $S_1, S_2, \dots, S_m$ such that there exists $i_1 \in S_2 \setminus S_1$, there exists $i_2 \in S_3 \setminus \{S_2 \cup S_1\}$,..., there exists $i_{m-1} \in S_m \setminus \{S_{m-1} \cup S_{m-2} \cup \ldots \cup S_1\}$. None of these sets is empty. For all l, let $g/S_l = \{ij, i \neq j, i \in S_l, j \in S_l\}$. As γ is parsimonious, $i_l \in S_l$ and not in other $S_{l'}$, $l' < l$ because of the ordering of the indexes. Let $g = \bigcup_{l=1}^{m} g/S_l$. The set g is a network. By parsimonious requirement 3.(i), there is no inclusion among such sets S_l. Thus, by construction, each such S_l is a clique of g and $\gamma_g = \gamma$. □

In the remainder of the section, we show that we can rationalize each network as an equilibrium outcome of the game we have developed in this paper.

Theorem 4 *The cover representation of each network is a stationary subgame perfect equilibrium outcome of a game in cover function form.*

Proof.

Let g be a network of n players in N. By Theorem 3, either $\gamma_g = \{N\}$, or the requirement 2 of the Definition 6 is true, or the requirements 3.(i) and 3.(ii) of the Definition 6 are true. For each case, we construct a triple (N, v, ρ), such that the equilibrium outcome of this game in cover function form is γ_g.

Part 1. Requirement 1

If $\gamma_g = \{N\}$, then define the cover function, v, by

$$v(N, \{N\}) > 0 \text{ and for all embedded coalition } (S, \gamma) \neq (N, \{N\}), v(N, \{N\}) = 0$$

Take an arbitrary protocol ρ. The stationary subgame perfect equilibrium outcome of this game is γ_g. In fact, the strategy consisting of proposing N as a proposer is sustained in equilibrium because other coalitions have a worth of zero and the discount factor tends to zero. The first player selected by ρ cannot do better than proposing to form the coalition N, with a division of the coalitional worth such that no player will reject (because of the discounting) and all the responders accept the proposal. Sequentially, all the following selected proposers cannot do better than making the same proposal.

Part 2. Requirements 3.(i) and 3.(ii)

We index all the players for identification. Consider the ordering of $\gamma_g \equiv \{S_1, S_2, \ldots, S_m\}$, such that there exists $i_1 \in S_2 \setminus S_1$, there exists $i_2 \in S_3 \setminus \{S_2 \cup S_1\}$,...,there exists $i_{m-1} \in S_m \setminus \{S_{m-1} \cup S_{m-2} \cup \ldots \cup S_1\}$.

Second, define ρ by $\rho^p(N) = i_0$ where i_0 a fixed player in S_1; $\rho^p(N \setminus \{i_0\}) = i_1$; $\rho^p(N \setminus \{i_0, i_1\}) = i_2$; ...; $\rho^p(N \setminus \{i_0, i_1, \cdots, i_{m-2}\}) = i_{m-1}$. Complete the selection of first proposers and $\rho^p(S) = i_S$ where i_S is the lowest indexed individual in S for any other subset $S \subseteq N$. Let $\rho^r(S)$ be the increasing ordering of indexed individuals in $S \setminus \{\rho^p(S)\}$ for all $S \subseteq N$.

Third, define the value function by $v(S, \gamma) = 0$ for each embedded coalition (S, γ) such that $S \notin \gamma_g$ or $\gamma \neq \gamma_g$, and $v(S_j, \gamma_g) = (m - j + 1)n$, for $j = 1, \ldots, m$.

Now we show that γ_g is a stationary subgame perfect equilibrium with for the game defined by (N, v, ρ). An equilibrium action sequence is:

At stage $k = 1$ of the game, $\rho^p(N) = i_0$ makes the first proposal. She can no do better than proposing S_1. In fact, suppose that i_0 proposes $S \notin \gamma_g$, this proposal will be rejected and the rejector will make another proposal in γ_g. Suppose that i_0 proposes another set in γ_g containing i_0, he can do better by proposing S_1 with the highest expected payoff. Following i_0, all the responders in S_1 accept the proposal. In case of rejection, S_1 will not form or will form with a delay, and this strategy is strictly dominated by accepting. Thus, S_1 forms and the updated set of proposers is $P_2 = N \setminus \{i_0\}$.

At stage $k = 2$ of the game, $\rho^p(P_2) = i_1$. Notice that $i_1 \in S_2 \setminus S_1$. For the same reasons as at stage $k = 1$, i_1 can not do better than making an acceptable proposal S_2.

The game continue the same way up to stage $k = m$ and S_m forms.

Now $k = m+1$, the remaining players in P_k cannot do better that confirming one of the previous coalitions they belong to and no rejection is possible. In fact the worth of any other coalition is zero.

Thus, no deviation from the previous strategy is profitable for the deviator. A stationary subgame perfect equilibrium outcome of the cover function (bargaining) game (N, v, ρ) is γ_g.

Part 3. Requirement 2

First, $|S| = 2$ for all $S \in \gamma$. Second, there exists a unique couple $S, S' \in \gamma$ such that $S \cap S' = \{i\}$, for all $i \in N$. We construct a game similar to the one in the Part 2 of the proof. There is a key difference here, which is $m = n$. For the protocol, everything is the same except that if S_1 up to S_{m-1} has formed, there remains only one proposer and he will make the last proposal. The justification to the equilibrium is the same, except that the last proposer here, will propose S_m instead of confirming an existing coalition because S_m yields extra payoffs. A stationary subgame perfect equilibrium outcome of the cover function (bargaining) game (N, v, ρ) is γ_g.

□

The result in Theorem 4 opens a new avenue for the understanding of coalitions and networks. More precisely, the economics literature focused only on the direction of the formation of coalitions following existence of networks. With the Theorem 4, it is now possible to explore the other direction. This is, how certain structures of coalitions can explain the formation of networks. For example, our model can rationalize the fact that participation to clubs creates bilateral links and can induce the formation of a network. Thus, the Theorem 4 closes a gap between network theory and coalition formation theory, and provides a new rationale for the sequential formation of networks.

5 Conclusion

The formation of coalitions have received a considerable attention in the economics theory. However, most of the studies are restricted to disjoint coalitions, and therefore, fail to address economic and social settings where coalitions may overlap. Our paper takes a step forward by proposing a new sequential game, namely a game in cover function form, with the specificity that the outcome is an overlapping coalition structure. More precisely, this game is not restricted to disjoint coalitions only, but also allows the same agent to belong to different coalitions. We discuss the existence of equilibria, and we develop an algorithm to compute equilibrium outcomes (subject to existence) for symmetric games in cover function form. Next, we contribute to the relation between networks and coalition structures by identifying the characteristics of overlapping coalition structures that make them uniquely represent networks. Owing to that, we define a one-to-one relation between networks and overlapping coalition structures. In the same logic, we propose a new way to rationalize the formation of networks, by showing that each one can be obtained as an equilibrium outcome of a game in cover function form.

This paper models the formation of overlapping coalitions, by a process of sequential offers with discounting, in the spirit of Rubinstein (1982) and Chatterjee et al. (1985). Agents sequentially propose to form a coalition and to share its worth. The coalition forms if the offer is unanimously accepted by its members. An interesting research direction is to investigate a situation where each player can propose as much coalitions as possible. Also, majority rule for agreement can be explored, as an alternative to unanimity. Furthermore, settings where the selection of proposers or respondents are random are also interesting to investigate. In fact, the topic is recent, and several open questions remain to be answered. We hope that our work will inspire other researchers for deeper investigations on the matter.

Appendix

Proof of Theorem 1.

Consider the game in cover function form, $\{N, v, \rho\}$ of n players. A coalition forms if and only if it is proposed by a player belonging to P_k at a stage k of the game. Thus, for the existence of an equilibrium, it is sufficient to focus on the set of proposers. Furthermore, notice that the size of P_k decreases (but not strictly) during the bargaining process. It decreases from n at the beginning of the game,to 0 at the end. Owing to that, we prove Theorem 1 by induction on the size of the set P_k.
At stage k of the game, $k \geq 1$, suppose that a collection of coalitions λ has formed. Let $\{P_k, v|_{\Gamma_\lambda}, \rho|_{\Gamma_\lambda}\}$ denote the game in cover function form such that

$v|_{\Gamma_\lambda} \equiv \{v(S,\gamma)_{S\in\gamma}\}_{\gamma\in\Gamma_\lambda}$ is the restriction of the cover function v to embedded sets (S,γ) such that $\gamma \in \Gamma_\lambda$, and

$\rho|_{\Gamma_\lambda} \equiv \{(\rho^p(P_k), \rho^r(S))\}_{\emptyset\neq S\in\gamma,\gamma\in\Gamma_\lambda}$.

This new game is nothing but the continuation of the game $\{N, v, \rho\}$ such that proposers have to be selected in P_k and the possible covers that will form are compatible with the already formed collection of coalitions, λ.

Let $M \equiv Max_{(S,\gamma)\in\Sigma}\{v(S,\gamma)\}$. Obviously, M exists because Γ is a finite set. If the game ends and a cover $\gamma \in \Gamma$ forms, the number of distinct coalitions in γ is at most n. The reason is that, a coalition forms following a proposal made by a player and there are n players. Besides, the cover function is nonnegative. Therefore, a payoff to and individual at the end of the game lies in the set $X \equiv [0, nM]$.

Consider the last node of the cover function game. At this stage, k_0, of the game, the size of P_{k_0} is 1. Note $P_{k_0} = \{i_0\}$. I affirm that player i_0 cannot make an unacceptable proposal. In fact, i_0 knows with certainty the cover that will form following his proposal. If he makes an unacceptable proposal, there will be a cost for the rejection and this will negatively affect the payoff of all the players, including himself. Therefore, the only case where he can do so and not being harmed is when he's overall maximum payoff is zero. In this case also, making an unacceptable proposal is ruled out the Assumption A.1.1. Thus, the best strategy for i_0, is to make an acceptable proposal that will yield the best payoff to him.

Assume that an equilibrium exists for each game in cover function form with less than n proposers.

Consider now the entire game $\{N, v, \rho\}$, with n proposers at the beginning. Let S be the first coalition that forms. We can make this assumption because, if there exists $(S,\gamma) \in \Sigma$ such that $v(S,\gamma) > 0$, at least one acceptable proposal will be made because, if bargaining continues forever, each player gains zero. Following the formation of S, the game moves to $\{P_k, v|_{\Gamma_{\{S\}}}, \rho|_{\Gamma_{\{S\}}}\}$ (as defined previously), with $n-1$ proposers. By assumption, since the size of P_k is less than n, an equilibrium exists for this game. Fix one equilibrium strategy for each player in this subgame. In the following, we describe equilibrium strategies after S has formed. For this purpose, we have to generate:

(i) A probability distribution β_S over the set $\Lambda_S \equiv \{\lambda, \lambda \cup \{S\} \in \Gamma_{\{S\}}\}$.

(ii) A set $\mathcal{U} \equiv \left\{\left(u^l(T)\right)_{l\in T}, T \in \lambda\right\}$ of equilibrium payoff vectors for all the players in the coalitions that are likely to form after S has formed. Notice that each $u^l(T) \in X$

At the beginning of the game, let $i \equiv \rho^p(N)$ be the first proposer. Let $\mathcal{N}^i \equiv \{S \subseteq N, i \in S\}$ denote the set of all coalitions containing i. Obviuosly, player i can only propose to form a coalition in $\mathcal{N}^i$. Let $\mathcal{A}^i \equiv \{\mathcal{N}^i \times \{j\}, j \in N \setminus \{i\}\}$, $\Delta^i \equiv \{$ Probability distribution over $\mathcal{A}^i\}$, and $\Delta \equiv \prod_{l\in N} \Delta^l$.

When selected as a proposer, player i has two options. He will either make an acceptable proposal to players in $S \in \mathcal{N}^i$ or make an unacceptable proposals to a player $j \in N \setminus \{i\}$. We can assume without loss of generally that $i \in N$ make unacceptable proposals only to a single player j. This is for simplicity but is equivalent to making a proposal to a set. The point is that, a rejection is a matter of only one player, the first rejector. A typical element $\alpha^i \in \Delta^i$ stands for the probabilistic choice for player i concerning the coalition to form following an acceptable proposal, or the player to whom an unacceptable proposal is to be made. Note $\alpha^i(S)$ the probability with which i chooses to make an acceptable proposal to $S \in \mathcal{N}^i$, and $\alpha^i\big(\{j\}\big)$ the probability with which i chooses to make an unacceptable proposal to $j \in N \setminus \{i\}$. Let x_S^l denote the equilibrium expected payoff to a player $l \in N$ that belongs to a coalition S, if i is the first proposer. Notice that ex-post, if S is not in the equilibrium cover, then $x_S^l = 0$. Thus, the overall equilibrium expected payoff, x^l, to a player l , if i is the first proposer, is the sum of what he receives from each coalition that he belongs to. Formally, $x^l = \sum_{S \in \mathcal{N}^l} x_S^l$.

Fix $\alpha \in \Delta$ and $x \equiv (x^l)_{l \in N} \in X^n$. The first proposer i has two options: make an acceptable proposal to a coalition $S \in \mathcal{N}^i$ or make a proposal that will be rejected by a player $j \neq i$.

Lemma 3 *If i makes an acceptable proposal to S, then her optimal expected payoff is* $g^i(S,x) \equiv \sum_{\lambda \in \Lambda_S} \beta_S(\lambda) v(S,\gamma) - \delta \sum_{j \in S; j \neq i} x^j$, *where* $\gamma = \{S\} \cup \lambda$.

Proof.

To alleviate the notation in the proof, we write (S,λ) instead of $\{S\} \cup \lambda$.

Player i names a coalition $S \in \mathcal{N}^i$ and makes an acceptable proposal $y_S(\gamma)$ conditional on $\gamma \in \Gamma_{\{S\}}$. He solves the following maximisation program:

$$Max_{y_S} \mathbf{E}\Big\{ y_S^i\big(S,\lambda\big) \Big\} \equiv \sum_{\lambda \in \Lambda_S} \beta_S(\lambda) y_S^i\big(S,\lambda\big) \tag{1}$$

Subject to:

$$\sum_{\lambda \in \Lambda_S} \beta_S(\lambda) y_S^j\big(S,\lambda\big) \geq \delta x^j \text{ for all } j \in S, j \neq i \tag{2}$$

$$\sum_{l \in S} y_S^l\big(\lambda\big) \leq v\big(S,(S,\lambda)\big) \tag{3}$$

The interpretations are the following.

(1): $i \in N$ maximizes her payoff expecting that the cover $\gamma = (S,\lambda)$ will form with probability $\beta_S(\lambda)$.

(2): each player $j \in S \setminus \{i\}$ accepts the offer made by i only if he is better off than what he could gain if he strategically rejects the proposal and becomes the next proposer.

(3): The aggregate payoff to all players in S can not be greater than the coalitional worth defined by the cover function v.

It is straightforward to see that inequalities (2) and (3) will be binding at equilibrium.
This is,

$$\sum_{\lambda \in \Lambda_S} \beta_S(\lambda) y_S^j\big(S,\lambda\big) = \delta x^j \text{ for all } j \in S, j \neq i \tag{4}$$

$$\sum_{l \in S} y_S^l\big(\lambda\big) = v\big(S,(S,\lambda)\big) \tag{5}$$

Sum (4) over j and add $\sum_{\lambda \in \Lambda_S} \beta_S(\lambda) y_S^i\big(S,\lambda\big)$. Let $g^i(S,x)$ denote the maximum value of the program. From (5), $g^i(S,x) = \sum_{\lambda \in \Lambda_S} \beta_S(\lambda) v\big(S,(S,\lambda)\big) - \delta \sum_{j \in S; j \neq i} x^j$. □

The functional, g^i, defined in the above Lemma 3, is a continuous function of x (as a sum of sums of projections and sums of continuous functions) and is independent of λ.
The player $i = \rho^p(N)$ will choose a coalition S in $\mathcal{N}_i$ and make an acceptable proposal to players in S if this coalition induces the highest $g_i(S,x)$ to him.

For each $l \in N$, let $v_l^i(x,\alpha)$ denote the expected payoff to player i if player l proposes at this stage. If the proposal made by player l is accepted, let B_l^i denote the equilibrium expected payoff to player i. If the proposal is rejected by some player j, let $\overline{j}$ denote the next proposer. According to the timing of the game, $\overline{j} = j$ if player j is a potential proposer ($j \in P_k$), else $\overline{j} \neq j$ and is selected by the protocol. Formally, we have

$$v_l^i(x,\alpha) = B_l^i + \delta \sum_{j \neq l} \alpha^l\big(\{j\}\big) v_{\overline{j}}^i(x,\alpha). \tag{6}$$

Lemma 4 *For each $l \in N$, v_l^i is a continuous function of (x,α).*

Proof.

Recall that the number B_l^i is defined for all $l \in N$. Since i is the first proposer, let us define this number for i separately and for the next proposer $\overline{j} \neq i$. We have

$$B_i^i \equiv \sum_{S \in \mathcal{N}^i} \alpha^i(S) g^i(S,x) \tag{7}$$

And for $j \neq i$

$$B_{\overline{j}}^i \equiv \delta \sum_{T \in \mathcal{N}^{\overline{j}}, i \in T} x_T^i \alpha^{\overline{j}}(T) + \sum_{T \in \mathcal{N}^{\overline{j}}, i \notin T} \alpha^{\overline{j}}(T) u^i(T) \tag{8}$$

The interpretation of these two equations are the following :
(7): With probability $\alpha^i(S)$, player i chooses a coalition S and makes an acceptable proposal.
(8): Player i's proposal is rejected, and a new proposal is made by $\overline{j} \neq i$ who makes an acceptable proposal to a coalition T. Either player i belongs to T, and we have the first expression, or not, and

we have the second expression.

The set of equations defining the functional v_l^i, for $l \in N$ can be defined as

$$V^i \equiv (v_l^i)_{l \in N} \ \text{, and } B^i \equiv (B_l^i)_{l \in N}$$

Let C denote the $n \times n$ matrix with 1's on the diagonals and $-\delta\alpha^l\big(\{j\}\big)$ as the $l\overline{j}$th off-diagonal element. Thus, $B^i = CV^i$.

In each row, the sum of the off-diagonal elements lies in $(-1; 0]$ and C is nonsingular.

Thus, $V^i = C^{-1}B^i$. We conclude that for each $l \in N$, v_l^i is a continuous function of x and α. Therefore, the equilibrium expected payoff to player i is a continuous function of x and α, whether his proposal is accepted or not. $\square$

For each $(x, \alpha) \in X^n \times \Delta$ and a fixed $i \in N$, define $v^i(x, \alpha, \cdot)$ on Δ^i by

$$v^i(x, \alpha, \alpha'^i) \equiv \sum_{S \in \mathcal{N}^i} \alpha'^i(S) g^i(S, x) + \delta \sum_{j \neq i} \alpha'_i\big(\{j\}\big) v_{\overline{j}}^i(x, \alpha)$$

and maximize this function with respect to $\alpha'^i \in \Delta^i$. Let $\phi_1^i(x, \alpha) \equiv Max_{\alpha'^i \in \Delta^i}\Big\{v^i(x, \alpha, \alpha'^i)\Big\}$, $\phi_2^i(x, \alpha) \equiv Argmax_{\alpha'^i \in \Delta^i}\Big\{v^i(x, \alpha, \alpha'^i)\Big\}$, and $\varPhi \equiv \prod_i \phi_1^i \prod_i \phi_2^i$.

Lemma 5 *$\varPhi$ is a mapping on $X^n \times \Delta$ and it admits a fixed point $(\overline{x}, \overline{\alpha})$.*

Proof.

According to the maximum theorem and the facts that $v^i(x, \alpha, \cdot)$ and $\phi_1^i(\cdot)$ are function, we have that $\phi_i^2(\cdot)$ is a convex-valued upper hemicontinuous correspondence. This result holds for all i in N and (x, α) in $X^n \times \Delta$ because i and (x, α) where chosen arbitrarily. Thus, $\prod_i \phi_1^i$ maps $X^n \times \Delta$ on X^n. Therefore, the correspondence $\varPhi$: $X^n \times \Delta \to X^n \times \Delta$ admits a fixed point $(\overline{x}, \overline{\alpha})$ according to Kakutani's fixed point theorem. $\square$

Lemma 6 *The fixed point of $\varPhi$ defines a stationary subgame perfect equilibrium strategy.*

Proof.

We denote by σ, the following strategy profile.

(i) For $P_k = N$, an arbitrary $i \in N$, as a proposer, makes proposals according to $\overline{\alpha}$:
  - to each coalition $S \in \mathcal{N}^i$ such that $\overline{\alpha}^i(S) > 0$, player i makes the proposal $y_S(\gamma)$ which solves the maximization problem addressed in the equations (1) – (3);

- to each $j \neq i$ such that $\overline{\alpha}^i(\{j\}) > 0$, player i offers less than $\delta\overline{x}^j_{\{i,j\}}$ (where j is in a coalition containing also i.

(ii) For $P_k = N$, and when $i \in N$ is selected to respond a proposal $y_S(\gamma)$, and such that every respondent j to follow i is offered an expected payoff of at least $\delta\overline{x}^j$. This is, $\sum_{\lambda \in \Lambda_S} \beta_S(\lambda) y^j_S(S,\lambda) \geq \delta x^j$ for all respondents j that follow i. Then, i accepts the proposal if and only if $\sum_{\lambda \in \Lambda_S} \beta_S(\lambda) y^i(S,\lambda) \geq \delta x^i$.

(iii) For $P_k = N$, and when $i \in N$ is a respondent. From (ii), if there is exactly one respondent to follow i, say $j \in N$, such that j is offered an expected value less than δx^j, then j will reject the proposal. Player i's decision will now depend on the present value of the payoff resulting from j rejecting the offer and this will lead to a new proposal as in (i). This value is precisely $\delta v^{i}_{\overline{j}}(\overline{x},\overline{\alpha})$. Player i accepts the proposal if and only if $\delta v^{i}_{\overline{j}}(\overline{x},\overline{\alpha}) \geq \delta x^i$.

  Now consider a proposal made to respondents $\{1,...,r\}$ in that order. Suppose we have computed the decisions of all respondents $i+1,...,r$. Player i's decision is then obtained by considering the decision of the next responder to reject the proposal, say j. Player i accepts the proposal if and only if $\delta v^{i}_{\overline{j}}(\overline{x},\overline{\alpha}) \geq \delta x^i$. In this way we obtain a complete description of the actions of all respondents of a proposal.

(iv) For $P_k \subset N$. It must be the case that some collection of coalitions λ has already formed. The strategies of the remaining players are defined according to the preselected equilibrium of the game $\{P_k, v|_{\Gamma_\lambda}, \rho|_{\Gamma_\lambda}\}$.

We show that any strategy profile, σ, satisfying (i)-(iv) is a stationary subgame perfect equilibrium. Consider σ and deviations that a single $i \in N$ can contemplate.
By construction $\overline{x}^i = v^i(\overline{x},\overline{\alpha},\overline{\alpha}^i) = Max\left\{v^i(\overline{x},\overline{\alpha},\cdot)\right\}$. Thus, it is not possible for i, as a proposer, to receive a higher payoff than $\overline{x}^i$ by making a one-shot deviation from $\overline{\alpha}^i$.
This implies that no other strategy can yield i a higher payoff than $\overline{x}^i$. The action prescribed in (i) achieves $\overline{x}^i$ and therefore cannot be improved upon. Suppose that i is a respondent and all respondents to follow i are offered at least $\delta\overline{x}^j$, which, by hypothesis, they will accept. By deviating, i gets a present value of $\delta\overline{x}^i$. Clearly then, the action prescribed in (ii) cannot be improved upon. Suppose i is a respondent who is followed by a respondent j who, based on σ, will reject the proposal. Accepting the proposal yields $\delta v^{i}_{\overline{j}}(\overline{x},\overline{\alpha})$ to $i \in N$ while rejecting it yields at most $\delta\overline{x}^i$. Thus, the action described in (iii) cannot be improved upon. A similar argument applies to the description in (iii) of i's actions in the other cases when i is a responder. Finally, notice that when some players have left the game, i can not do better than the actions in (iv). Thus, σ is a stationary subgame perfect equilibrium of the game in cover function form $\{N, v, \rho\}$. $\square$

The induction ends with the result obtained in the lemma above. The strategy profile σ is then a stationary subgame perfect equilibrium where the only source of mixing is in the choice of a coalition by each proposer. $\square$

Proof of Theorem 2.

We proceed by a series of two lemmas. The first one shows the existence of δ^* and the second one establishes conditions to be verified by equilibrium payoffs. We conclude by an induction argument on the size of P_k.

Lemma 7 *There exists $\delta^* \in (0,1)$ such that for all $\delta \in (\delta^*, 1)$, and all representative overlapping coalition $\boldsymbol{r}$ at a stage k_r of the game such that $|P_{k_r}| \geq 1$, there is a unique set $\{s; \boldsymbol{o}\}_{\boldsymbol{r}}$ that maximizes $\frac{v\left(\{s;\boldsymbol{o}\}, c_{\mathbf{r}\bullet\{s;o\}}\right)}{1+\delta(s-1)}$.*

Proof.

Suppose that the game is a the stage k with $|P_k| = 1$. We have already shown (in the proof of Theorem 1) that the best option to this last proposer is to make an acceptable proposer. Therefore, he proposes the coalition S such that the corresponding representative overlapping coalition $\{s; \mathbf{o}\}^*$ maximizes the average payoff that is uniquely selected by the algorithm. Thus, $\{s; \mathbf{o}\}^*$ is the unique maximizer of $\frac{v\left(\{s;\mathbf{o}\}, c_{\mathbf{r}\bullet\{s;o\}}\right)}{s}$. We show later in the proof that $\{s; \mathbf{o}\}^*$ is also the only one maximizer of $\frac{v\left(\{s;\mathbf{o}\}, c_{\mathbf{r}\bullet\{s;o\}}\right)}{1+\delta(s-1)}$.

Consider a stage of the game such that $P_k > 1$ and fix a representative overlapping coalition structure $\mathbf{r}$ corresponding to the collection of coalitions that have already form a the stage k of the game. Now, consider the sequence $\{\delta^q\}$ in $(0,1)$ such that $\delta^q \to 1$. Let $\mu(\mathbf{r}, \delta^q) \equiv Argmax_{\{s;\mathbf{o}\}} \left\{ \frac{v\left(\{s;\mathbf{o}\}, c_{\mathbf{r}\bullet\{s;o\}}\right)}{1+\delta^q(s-1)} \right\}$. By the maximum theorem, this correspondence is upper hemicontinuous. Since the set of maximizers is finite, there exists δ^n such that $\mu(\mathbf{r}, \delta^q) \subseteq \mu(\mathbf{r}, 1)$ for all $\delta^q \geq \delta^n$. We proceed in the same way for all possible $\mathbf{r}$ at this stage and we have a number δ^n for each. By taking the maximum of these δ^ns, we state that there exists δ^* such that for such $\mathbf{r}$, $\mu(\mathbf{r}, \delta^q) \subseteq \mu(\mathbf{r}, 1)$ for all $\delta^q \geq \delta^*$. Say differently, if $\delta \geq \delta^*$, then for all such $\mathbf{r}$, the maximizer $\{s; \mathbf{o}\}^*_{\mathbf{r}}$ of $\frac{v\left(\{s;\mathbf{o}\}, c_{\mathbf{r}\bullet\{s;o\}}\right)}{1+\delta(s-1)}$ maximizes also $\frac{v\left(\{s;\mathbf{o}\}, c_{\mathbf{r}\bullet\{s;o\}}\right)}{s}$ which is the average value used in the algorithm. Let $M_{\mathbf{r}}$ denote the maximum of the average value. It remains to show that if $\delta \geq \delta^*$, then $\mu(\mathbf{r}, \delta)$, contains only one such $\{s; \mathbf{o}\}^*$, and that the corresponding s^* is the largest possible, that maximizes the average worth. This is, $\mu(\mathbf{r}, \delta)$ is a singleton set, $\left\{\{s; \mathbf{o}\}_\delta\right\}$.

Obviously, for $|P_k| = 1$, $\{s; \mathbf{o}\}^*$ is unique and $\emptyset \neq \mu(\mathbf{r}, \delta) \subseteq \mu(\mathbf{r}, 1)$. Thus, $\{s; \mathbf{o}\}^* = \{s; \mathbf{o}\}_\delta$.

For $|P_k| > 1$, suppose the contrary. This is, there exists $\delta \geq \delta^*$ and $\{s; \mathbf{o}\}^*$ such that $s^* < s_\delta$. For $s^* < s_\delta$, $\frac{1-\delta}{s_\delta} + \delta < \frac{1-\delta}{s^*} + \delta$. By assumption A.2, $M_{\mathbf{r}}$ is positive. Thus, $\frac{s_\delta M_{\mathbf{r}}}{1+\delta\left(s_\delta - 1\right)} > \frac{s^* M_{\mathbf{r}}}{1+\delta(s^*-1)}$. Hence,

$\frac{v\left(\{s;\mathbf{o}\}_\delta, c_{\mathbf{r}\bullet\{s;\mathbf{o}\}_\delta}\right)}{1+\delta(s_\delta-1)} > \frac{v\left(\{s;\mathbf{o}\}^*, c_{\mathbf{r}\bullet\{s;\mathbf{o}\}^*}\right)}{1+\delta(s^*-1)}$ and this is a contradiction, because $\{s;\mathbf{o}\}^*$ is a maximizer of $\frac{v\left(\{s;\mathbf{o}\}_\delta, c_{\mathbf{r}\bullet\{s;\mathbf{o}\}_\delta}\right)}{1+\delta(s_\delta-1)}$. This concludes the prof of Lemma 8. □

Lemma 8 *At the stage k of the game such that $|P_k| \geq 1$, let $(x^l)_l$ denote the collection of equilibrium expected payoffs.*

If player $i \in P_k$ makes an acceptable proposal to coalition S with a positive probability, then

(i) $(j \in S, j \neq i$ and $x^l < x^j) \Longrightarrow l \in S$

(ii) $x^i \leq x^l$ for all $l \notin P_k$

Proof.

Notations here refer to the notations that we have introduced previously in the proof of Theorem 1. Suppose that player $i \in P_k$ makes an acceptable proposal to S of size s and overlapping status $\mathbf{o}(S)$. Let $c_{\mathbf{r}\bullet\{s;\mathbf{o}(S)\}}$ be the resulting overlapping representative coalition structure.

By the proof of Theorem 1, $x^i \geq v\big(\{s;\mathbf{o}(S)\}, c_{\mathbf{r}\bullet\{s;\mathbf{o}(S)\}}\big) - \delta\sum_{j\in S, j\neq i} x^j$, and this is not less than $Max_{T\in\Gamma_\lambda, i\in T}\left\{v\big(\{|T|;\mathbf{o}(T)\}, c_{\mathbf{r}\bullet\{|T|;\mathbf{o}(T)\}}\big) - \delta\sum_{j\in T, j\neq i} x^j\right\}$. This result holds because S is the best offer that i can make at this stage.

Therefore, $l \notin S$ implies that $x^l \geq x^j$. Otherwise i can propose a set T containing l, such that $|T| = |S|$ and $\mathbf{o}(T) = \mathbf{o}(S)$. By doing so, he can replace j by l and increases his payoff. Thus, by contraposition, we obtain the first part (i) of the lemma.

Suppose that result (ii) of the lemma is false. This is, there exists $l \in P_k$ such that $x^i > x^l$.

If $l \notin S$ then player l can form a coalition of the same size as S and the same overlapping status, by replacing player i by herself. By doing so, player l will receive the same payoff x_i because of the symmetric cover function and this is a contradiction to (i) .

Now suppose that $l \in S$. Thus, $x^l \geq v\big(\{s;\mathbf{o}(S)\}, c_{\mathbf{r}\bullet\{s;\mathbf{o}(S)\}}\big) - \delta\sum_{j\in S, j\neq l} x^j = v\big(\{s;\mathbf{o}(S)\}, c_{\mathbf{r}\bullet\{s;\mathbf{o}(S)\}}\big) - \delta\sum_{j\in S, j\neq i} x^j + \delta x^l - \delta x^i$.

Then using the previous inequality

$$v\big(\{s;\mathbf{o}(S)\}, c_{\mathbf{r}\bullet\{s;\mathbf{o}(S)\}}\big) - \delta\sum_{j\in S, j\neq i} x^j \geq Max_{T\in\Gamma_\lambda, i\in T}\left\{v\big(\{|T|;\mathbf{o}(T)\}, c_{\mathbf{r}\bullet\{|T|;\mathbf{o}(T)\}}\big) - \delta\sum_{j\in T, j\neq i} x^j\right\}$$

Thus

$x^l \geq x^i$ which is a contradiction.

Therefore, for all $l \in P_k$, $x^l \geq x^i$ and the second part (ii) of the lemma is true. □

Remaining proof of Theorem 2.

Fix an equilibrium as described in the theorem and let δ lies in $(\delta^*, 1)$ with δ^* defined as in the

Lemma 8. We proceed by induction on the size of P_k, following the departure of a collection of coalitions λ.

If $P_k = \{i\}$, then player i will make an acceptable proposal to the players that insure him the highest payoff.

Suppose by induction that the theorem holds at any stage where $|P_k| = 1, 2, \cdots, m$ for some $m \leq n$.

Consider now a stage where $|P_k| = m + 1$. We prove that the next coalition S to be proposed admits the representative overlapping coalition structure $\{s; \mathbf{o}\}^*$ as proposed in the algorithm. Since each player in P_k makes an acceptable proposal to some coalition with positive probability, it comes from induction hypothesis and result (ii) of the Lemma 9 that $x^i = x^j = x$ for all $i, j \in P_k$. It follows from the induction and the optimality of the proposal that

$x = v\big(\{s; \mathbf{o}(S)\}, c_{\mathbf{r}(\lambda)\bullet\{s;\mathbf{o}(S)\}}\big) - \delta(s-1)x \geq v\big(\{|T|; \mathbf{o}(T)\}, c_{\mathbf{r}(\lambda)\bullet\{|T|;\mathbf{o}(T)\}}\big) - \delta(|T|-1)x$

for all $T \ni i$. This implies that $\{s; \mathbf{o}(S)\} \in \mu(\mathbf{r}, \delta)$. And we conclude by Lemma 8 that $\{s; \mathbf{o}(S)\} = \{s; \mathbf{o}\}^*$.

Of course the payoff to a proposer is $M_{\mathbf{r}}(\delta)$. $\square$

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