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A recursive core for cooperative games with overlapping coalitions

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Abstract This paper develops an extension of the recursive core to the setting of overlapping coalitions. We show that the cooperative game theoretical traditional way of separating a deviant coalition from the game played by the ones left behind is no more satisfactory. We therefore introduce a new paradigm with which we obtain the overlapping coalition structure core whose allocations are Pareto-efficient.

Keywords: Overlapping coalitions; Cover function game; Recursive core; Residual game; Optimistic; Pessimistic; Efficiency.

JEL Classification: C70, C71, D71

1 Introduction

The core was developed by Edgeworth in 1881 and redefined in the vocabulary of the game theory by Gillies in 1959. It is the most popular solution concept used to assess the stability in coalition formation games. The concept was first defined for characteristic function games, and thereafter extended to partition function games (Thrall and Lucas, 1963) where externalities are taken into account. This is, in the event that a coalition of players deviates from a prescribed agreement, the payoff they expect to obtain depends on the way the remaining (residual) players react to the deviation.[1] Different extensions lean towards one way or another to model the behaviour of such residual players, leading to the $\alpha-$ and $\beta-$core (Aumann and Peleg, 1960), the $\omega-$core (Shenoy, 1979), the $\delta-$core (d'Aspremont et al., 1983), and the $\gamma-$core (Tulkens and Chander, 1997) to name but few. While the first notion is pessimistic and anticipates that the residual players will act to minimize the worth of the deviant coalition, the second is optimistic and requires its maximization. According to the third notion, the residual players will stick together but the forth anticipates their disintegration into

[1] In characteristic function games, a payoff to a coalition that deviates from an agreement is independent of the reaction of the residual players. For a good description of the core in characteristic function games, see Serrano (2007).

singletons.[2] As one can notice, each of these extensions is sensitive to a strong behavioral assumption on the residual players. But if such residual players are rational, should they not just maximize their own payoff and therefore play a game consistent to the one that leads to the deviation at the first place? Huang and Sjostrom (2003) discuss such consistency for several solution concepts. In the same logic, Koczy (2007) develops the recursive core, an extension of the coalition structure core introduced by Greenberg (1994), and shows that it is less sensitive to behavioral assumptions. That being said, none of these extensions of the core to partition function games, including the recursive core, addresses the case of overlapping coalitions. Yet, more than often, coalitions overlap in many economic issues and here are some. Many countries sign free trade agreements within overlapping coalitions of countries and more than half of the world trade is conducted under such agreements. Also, environmental agreements become widespread (due to climate change) in all areas of the globe and such agreements involve overlapping coalitions of countries. In developing countries, informal insurance occurs within overlapping groups of villagers (van den Brink and Chavas, 1997). In India, overlapping coalitions of political parties have participated in the 2005 state elections in Jharkhand (Bandyopadhyay and Chatterjee, 2006). These examples show that coalitions of economic agents naturally overlap when they strategically interact, and the game theory should pay more attention to overlapping coalitions. [3]

This paper develops an extension of the recursive core to the setting of overlapping coalitions. We first follow the tradition in cooperative game theory, defining the set of residual players as the complement of the set of a deviant coalition. This benchmark induces the *generalized recursive core* which coincides with the recursive core in the case of disjoint coalitions. However, properties of the generalized recursive core allocations are not fully satisfactory. We respond to this weakness by proposing a new paradigm where the set of residual players can intersect the one of the deviant coalition. For this purpose, we identify two types of players, *active* and *passive*, and only the later can belong to that intersection. We then obtain the *overlapping coalition structure core*. We show that this new notion is a refinement of the generalized recursive core and that the core allocations are Pareto-efficient.

Even though the topic of overlapping coalitions have been widely studied in computer science and robotics (Kraus et al. 1998, Lin and Hu 2007, and Dang et al. 2006), it is less addressed the game theory. To the best of our knowledge, Boehm (1973) is the first to study the endogenous formation of overlapping coalitions structures in the light of his general equilibrium model, followed by Myerson

[2] A reader who is interested in the difference between notions of the core will find an interesting overview in Hafalir (2007).

[3] Ray (2007) also suggest in his book on coalition formation games that the game theory should explore more the setting of overlapping coalitions.

(1980) who investigates allocation rules in such setting. More recently, Albizuri et al. (2006) develop an extension of the Owen value, and therefore the Shapley value, to overlapping coalitions. In their paper, the number of distinct coalitions that an individual can belong to is limited. Chalkiadakis et al. (2008, 2010) propose an extension of the core to overlapping coalitions. In their framework, a coalition structure (with possibly infinite coalitions) is a list rather than a set. Agbaglah (2014) develops a sequential game for overlapping coalitions but on a noncooperative perspective. This paper differs from all the previous ones because it is the first to develop an extension of the recursive core, and consequently the coalition structure core, to the setting of overlapping coalitions.

The remainder of the paper is organized as follows. We set the ingredients up in Section 2 through basic definitions, and we develop our benchmark extension in Section 3 and discuss its implications. In Section 4 we propose the main extension of the recursive core and compare the two notions. We conclude the paper in Section 5.

2 Ingredients

2.1 Background: the setting of partitions

Let N be a set of n players. A *coalition* S is a non-empty subset of N. A *partition* of N, π, is a collection of coalitions, $S_1, S_2, \ldots, S_m$, such that

- for all $i, j, \in \{1, 2, \cdots, m\}$, $\left(i \neq j \ \Rightarrow S_i \cap S_j = \emptyset \right)$, and
- $\bigcup_{i=1}^{m} S_i = N$

A partition is therefore a covering of N by disjoint coalitions. We denote by Π the set of all partitions of N. For each subset of N, $S \subsetneq N$, we denote by π_S a partition of S.

A *characteristic function* is a functional $w : 2^N \rightarrow \Re$.

A *partition function* is a functional $v : \Pi \rightarrow \left(2^N \rightarrow \Re\right)$ that assigns a characteristic function to each partition of N. A *partition function game* is the pair (N, v), where N is a set of players and v a partition function.

We borrow the following definitions from Koczy (2007).

1. Outcome

   An *outcome* of the partition function game (N, v) is a pair (x, π) with $x \in \Re^N$ and $\pi \in \Pi$ such that (i) $x_i \geq 0$ and (ii) $x(S) = v(S)$ for all $S \in \pi$, where $x(S) = \sum_{i \in S} x_i$.
2. Residual game

   Consider a set $R \subsetneq N$. If $N \setminus R$ have committed to form a partition $\pi_{N \setminus R}$, then the *residual game*, $(R, v_{N \setminus R})$ is a partition function game such that $v_{N \setminus R}(C, \pi_R) = v(C, \pi_R \cup \pi_{N \setminus R})$ for each $C \in \pi_R$.

3. Recursive core (optimistic, pessimistic)

   Let (N, v) be a partition function game.

   (i) The core of $\big(\{1\}, v\big)$ is the only outcome with the trivial partition: $C\big(\{1\}, v\big) = \Big\{v\big(1, (1)\big), (1)\Big\}$.

   (ii) Assume that the core $C(R, v)$ has been defined for all partition function games with at most $k-1$ players. Define $A(R, v) \equiv C(R, v)$ if $C(R, v) \neq \emptyset$, and $A(R, v) \equiv \Omega(R, v)$ otherwise, where $\Omega(R, v)$ is the set of outcomes in the game (R, v).

   (iii) The outcome (x, π) is dominated via the coalition S forming a partition π_S if for at least one (respectively all) $(y_{N\setminus S}, \pi) \in A(N \setminus S, v_{\pi_S})$, there exists an outcome $\big((y_S, y_{N\setminus S}), \pi_S \cup \pi_{N\setminus S}\big) \in \Omega(N, v)$ such that $y_S > x_S$. The outcome (x, π) is dominated if it is dominated via a coalition.

   (iv) The *optimistic (respectively pessimistic) core* is the set of non-dominated outcomes.

This pair of recursive cores is less sensitive to behavioural assumptions of optimism or pessimism than the existing notions of core (Koczy, 2007). The aim of this paper is to extend the recursive core to the broader and more realistic setting of overlapping coalitions.

2.2 Some definitions

A cover of N, γ, is a collection of coalitions, $S_1, S_2, \ldots, S_m$, such that $\bigcup_{k=1}^{m} S_k = N$.[4] We denote by Γ the set of all covers of N. Similarly, for each coalition S we denote by γ_S a cover of S, and Γ_S the collection of all such covers. For each collection of coalitions λ, we say that a cover γ is *compatible* with λ if $\lambda \subseteq \gamma$, and we denote by $\Gamma(\lambda)$ the collection of all such covers.

In the remainder of the paper, we use the term of *overlapping coalition structure* at the place of cover, whenever we emphasize on its composition.

An *embedded coalition*[5] is a pair (S, γ) such that $S \in \gamma$ and $\gamma \in \Gamma$. Finally we denote by Σ the collection of all embedded coalitions.

Definition 1 A cover function $V : \Sigma \rightarrow \Re$ assigns to each embedded coalition (S, γ), a nonnegative value $V(S, \gamma) \geq 0$.[6] A *cover function game* (***CFG*** in the remainder of the paper) is the pair (N, V), where N is the set of players, and V a cover function.

In the setting of overlapping coalitions, a player i may belong to more than one coalition and the total gain for i, x_i, is defined as the summation of the payoffs he gains from all individual coalitions he belongs to. Formally, if x_i^S is the payoff that player i receives from coalition S to which he belongs,

[4] Notice that each partition is a cover, but with the constraint that coalitions are disjoint.

[5] We borrow this definition from Macho-Stadler et al. (2007).

[6] In the definition of a cover function, we normalize the value function to be nonnegative (as it is usually done in this strand of the literature). All our results remain true as far as we impose any lower bound to the cover function. Therefore, the choice of zero is not a limitation but is done only for simplicity.

then the (total) *payoff* awarded to player i is $x_i \equiv \sum_{S \ni i} x_i^S$. This definition of the total payoff has a sense because, by definition, the *CFG* is a transferable utility game. The vector of all individual payoffs from S is denoted by $x_S \equiv (x_i)_{i \in S}$. An *outcome* of the *CFG* (N, V), is an ordered pair (x, γ) where $x \in \Re^n$, and $x \equiv (x_i)_{i \in N}$, such that for all $S \in \gamma$, $\sum_{i \in S} x_i^S = V(S, \gamma)$. We denote by $\Omega(N, V)$ the set of all outcomes in the *CFG* (N, V).

Definition 2 Given a *CFG* (N, V), a player $i \in N$ is an *overlapping player* if there exists a cover γ, and two distinct embedded coalitions (S, γ) and (S', γ) such that

- $i \in S \cap S'$, and
- $V(S, \gamma) > 0$ and $V(S', \gamma) > 0$.

We denote by O_S the set of all overlapping players in a coalition S.

The Definition 2 above is guided by the intuition that an overlapping player can impact the outcome of the game, when he chooses to belong to more than one coalition. Such players are very important in our setting, since banning them makes our framework equivalent to the setting of partitions.

2.3 Deviation

The meaning of deviation in a setting where coalitions may overlap is far from being obvious. Traditionally in cooperative games, a deviation from an existing coalition structure leads to a reconfiguration of the partition of players and inversely. However, this reconfiguration takes various non-traditional forms in our setting. We identify two distinct elementary forms and define each deviation as their combinations.

The first elementary form, *partial deviation*, is the formation of a new coalition without breaking existing ones. An example is a deviation from $\gamma = \{S_1, S_2\}$ that leads to the formation of $\gamma' = \{S_1, S_2, S_3\}$. This may happens anytime that some players in S_1 and S_2 decide to form a new coalition S_3 without breaking there original coalitions. Necessarily, the distribution of payoffs will be affected by such move, otherwise the new structure presents no interest for analysis. The second elementary form, *total deviation*, is the formation of a new coalition with a dislocation of existing ones. This is more familiar in the setting of partitions. An example is a deviation from $\gamma = \{S_1, S_2\}$ that leads to the formation of $\gamma'' = \{S_3, S_4\}$, with $\gamma'' \neq \gamma$.

We propose the following example to help the reader fix ideas before we move to the formal definition of a deviation. For simplicity of notation, we write a coalition explicitly $S = ijk...$ instead of $S = \{i, j, k, ...\}$.

Example 1 Consider the following 5-person *CFG*.
$N = \{i, j, k, l, m\}, \gamma = \{ij, kl, jm\}, V(ij, \gamma) = 1, V(jm, \gamma) = 10$, and
$V(S, \gamma') = 0$ for any other $(S, \gamma') \in \Sigma$.
Player j is an overlapping player in the game (N, V) because, $V(ij, \gamma) > 0$ and $V(im, \gamma) > 0$.
Consider the following covers $\gamma' = \{ij, klm\}$, $\gamma'' = \{ij, kl, m\}$, $\gamma''' = \{ij, kl, jm\}$.
In a deviation from γ' to γ'', player m performs a total deviation because he leaves the coalition klm and this coalition is no more in γ'. In contrast, considering a deviation from γ'' to γ''', player j performs a partial deviation. The coalition ij that contains player j is maintained in the new cover γ'''. Notice that this partial deviation is performed by player j who is an overlapping player.

Definition 3 Let $\gamma, \gamma' \in \Gamma$ and consider a deviation from γ to γ'.

1. **Total deviation** (individual deviation that can be performed by any player).
   A player $i \in N$ performs a total deviation if and only if there exists an embedded coalition (S, γ) such that $i \in S$ and $S \notin \gamma'$.[7]
2. **Partial deviation** (individual deviation that can only be performed by an overlapping player).
   A player $i \in N$ performs a partial deviation if and only if i is an overlapping player and for all embedded coalition (S, γ), if $i \in S$, then $S \in \gamma'$.
3. We use the term **deviation** to encompass partial and total deviations.
4. **Set deviation**.
   A player set $D \subseteq N$ deviates to form a cover γ_D of D if and only if each player $i \in D$ performs a deviation.

The set deviation is a collection of multiple individual deviations, some are total and others are partial. A deviation may affect the payoffs of all players, including non-deviators. Due to these externalities, if an overlapping player partially deviates from a single coalition to more than one coalition, the payoff of the players in the former coalition may be affected. But, he decides to deviate only if his overall payoff from the new cover (obtained after deviation) is higher than previously.

2.4 Residual game

Since a deviation may affect the payoffs of all players, if a player set $D \subseteq N$ deviates, the remaining players in $N \setminus D$, *residual players*, may react to this deviation. They will also play a game, similar to the stage game, called the *residual game*. This consistency requirement, introduced in the definition of the recursive core makes this notion of the core very attractable for us while considering the

[7] All coalitions that contain some total deviating players break up in the new cover γ'. This is the traditional notion of deviation.

extension of the core to overlapping coalitions.

Traditionally in cooperative games, a residual game concerns only the complement of the deviating set of players, reason why it is called residual. We stick to this tradition for a benchmark definition of the residual game for overlapping coalitions. However, later in the paper, we propose another definition that diverges from the tradition but suits better to our setting.

Suppose that a player set $D \subseteq N$ performs a deviation forming of a cover γ_D of D. Let $R \equiv N \setminus D$ be the set of residual players. Consider embedded coalitions of the form (S, γ_R) where $S \subseteq R$ and $\gamma_R \in \Gamma_R$ is a cover of R. Let Σ_R be the collection of all such embedded coalitions.

Definition 4 The residual game is the *CFG* (R, V_{γ_D}) where V_{γ_D} is such that for all $(S, \gamma_R) \in \Sigma_R$, $V_{\gamma_D}(S, \gamma_R) \equiv V(S, \gamma_R \cup \gamma_D)$.

Notice that in the Definition 4 above, $\gamma_R \cup \gamma_D$ is a cover of N, and $(S, \gamma_R \cup \gamma_D) \in \Sigma$ is an embedded coalition of N. Therefore the residual game is well defined and is a *CFG* on its own. This definition implicitly says that once players deviate and form a new cover, they commit to this cover. There is no room for a subsequent renegotiation. This assumption, also made by Koczy (2007), may sound restrictive but it is commonly made for practical reasons so that recursion can be defined. In fact, we need this assumption to build the recursion as the residual game is a game on its own, defined on the complement of S, with a **fewer** number of players than the initial game.

Notice finally that, if we ban overlapping players from our setting (if we consider only partitions instead of covers), then the Definition 4 is equivalent to the one in the framework of partition functions (that we have recalled in the section 2.1).

2.5 Dominance

Roughly speaking, an outcome is dominated if some players can deviate from it and obtain better payoffs. To take into account all possible reactions that can follow a deviation, we define two notions of dominance, the *optimistic dominance* and the *pessimistic dominance*. For this purpose we use the following partial order relation in $\Re^n$. For $x, x' \in \Re^n$ $\left(x = (x_i)_{i \in N}\right.$ and $\left.x' = (x'_i)_{i \in N}\right)$ and $S \subseteq N$, $x >_S x'$ if $x_i \geq x'_i$ for all $i \in S$, and $x_i > x'_i$ for some $i \in S$.

Definition 5 Optimistic (respectively **pessimistic**) **dominance**

An outcome (x, γ) is dominated via a coalition S forming a cover γ_S of S, if for **at least one** outcome (respectively **all** outcomes) (x'_R, γ_R) from the residual game, with $R \equiv N \setminus S$, there exists an **outcome** $\big((x'_S, x'_R), \gamma'\big)$, with $\gamma' \equiv \gamma_S \cup \gamma_R$, such that $x' >_S x$.

From an optimistic perspective, a player set may deviate as soon as there is one possibility to improve the payoffs of its players. This is, as soon as there exists an outcome from the residual game that make the deviators better: even if it is not sure residual players will allow that. On the other hand, in the pessimistic dominance, a player set deviates only if it is certain that its players will be better off. In this case, deviation occurs independently of the strategy played by the residual players. Obviously, while the optimistic dominance is very easy to obtain, the pessimistic one is very hard.

2.6 Generalized recursive core

In this section we generalize the recursive cores (Koczy, 2007) to overlapping coalition structures. Suppose that discussion commences from the grand coalition $\{N\}$. The *CFG* (N, V) awards a payoff to each player corresponding to the formation of the grand coalitions. From there, a player set S may be better off by deviating and forming a cover γ_S. Following this deviation, the remaining players may react strategically by playing the residual game. Remember that once γ_S forms, players in S commit to it.

Consider the *CFG* (N, V), and let $n \equiv |N|$. We define the core recursively on the number of players.

(i) For a game with only one player, say $\{1\}$, the core is $C(\{1\}, V) \equiv \big\{V(\{1\}, \{1\}); \{1\}\big\}$.

(ii) Assume that the core $C(R, V)$ is defined for every k-person *CFG* (R, V), $k = 1, 2, \ldots, n-1$. If the core is empty, define $A(R, V) \equiv \Omega(R, V)$, otherwise, define $A(R, V) \equiv C(R, V)$.

(iii) Now consider the *CFG* (N, V). An outcome (x, γ) is dominated via a coalition S forming a cover γ_S of S, if for **at least one** outcome (respectively **all** outcomes) $(x'_R, \gamma_R) \in A(R, V_{\gamma_S})$, with $R \equiv N \setminus S$, there exists an **outcome** $\big((x'_S, x'_R), \gamma\big)$, with $\gamma \in \Gamma(\gamma_S)$, such that $x' >_S x$.

(iv) The **generalized optimistic** (respectively **pessimistic**) **recursive core** of an n-person game, $C_+(N, V)$ $\big($respectively $C_-(N, V)\big)$, is the set of non-dominated outcomes by optimistic (respectively pessimistic) dominance.

Example 2 We denote by (N, V) the following 3-person cover function game, with $N = \{i, j, k\}$. We use the following Table 1 to summarize the value function.

The first thing to notice is that k is an overlapping player. In this example, the generalized recursive cores are not empty. For example the outcomes $z_1 = \big((1, 1, 2), \gamma_3\big)$, $z_2 = \big((1, 1, 2), \gamma_4\big)$ and $z_3 = \big((2, 2, 4), \gamma_6\big)$ are not dominated (pessimistic and optimistic). Thus they belong to $C_+(N, V)$ and

Table 1: An example of 3-person *CFG*

Covers	Cover function
$\gamma_1 = \{N\}$	$v(N, \gamma_1) = 1$
$\gamma_2 = \{i, j, k\}$	$v(i, \gamma_2) = v(j, \gamma_2) = 2$ and $v(k, \gamma_2) = 2$
$\gamma_3 = \{ik, j\}$	$v(ik, \gamma_3) = 3$ and $v(j, \gamma_3) = 1$
$\gamma_4 = \{i, jk\}$	$v(i, \gamma_4) = 1$ and $v(jk, \gamma_4) = 3$
$\gamma_5 = \{ij, k\}$	$v(ij, \gamma_5) = 0$ and $v(k, \gamma_5) = 2$
$\gamma_6 = \{ik, jk\}$	$v(ik, \gamma_6) = 4$ and $v(jk, \gamma_6) = 4$
For any other cover γ	$v(S, \gamma) = 0$

$C_-(N, V)$. Notice in the Example 2 that the two notions of the generalized recursive cores coincide because of the few number of players.

In this following, we derive some properties of the generalized recursive core.

Proposition 1 *The generalized optimistic and pessimistic recursive cores coincide with these notions on partition function games.*

Proof.

In partition function games, coalitions cannot overlap. Therefore, consider the cover function games (N, V) such that for all cover $\gamma \in \Gamma$, and all pairs of distinct embedded coalitions (S, γ) and (S', γ), $S \cap S' \neq \emptyset \Rightarrow V(S, \gamma) = V(S', \gamma) = 0$.

For all cover $\gamma = \{S_1, S_2, ..., S_m\}$. Suppose without loss of generalities that the l first coalitions do not intersect and let $S'_{l+1} \equiv \bigcup_{j=l+1}^{m} S_j$. Let $\pi_\gamma = \{S_1, S_2, ..., S_l, S'_{l+1}\})$. Thus, π_γ is a partition of N. Define for all $S \in \pi_\gamma$, $V'_\gamma(S, \pi_\gamma) = V(S, \gamma)$ if $S \neq S'_{l+1}$ and $V'_\gamma(S'_{l+1}, \pi_\gamma) = 0$. Thus, (N, V') defines a partition function game which is the restriction of the *CFG* (N, V) to partitions (since the values functions are nonnegative). By definition, it is obvious that if an outcome (x, γ) is non-dominated in the *CFG* (N, V), the corresponding outcome (x', π_γ) is non-dominated in the partition function game (N, V'). Thus, the generalized recursive core coincides with the recursive core. □

Proposition 2 *For a* CFG (N, V)*,* $C_+(N, V) \subseteq C_-(N, V)$

Proof.

We proceed by induction on the number of players.

For only one player,$|N| = 1$, $C_+(N, V) = C_-(N, V)$, trivially by the definition.

Assume that for a k-person *CFG* (N, V), $k < n$, $C_+(N, V) \subseteq C_-(N, V)$.

Now consider the game (N, V) with $|N| = n$.

If $C_+(N, V) = \emptyset$, then it is trivially included in $C_-(N, V)$

If $C_+(N,V) \neq \emptyset$, let (x,γ) be an outcome in $C_+(N,V)$, and suppose that (x,γ) is not in $C_-(N,V)$. If $(x,\gamma) \notin C_-(N,V)$, then there exists a coalition S such that (x,γ) is dominated (pessimistic) via S, forming γ_S. Thus, for **all** outcome (x'_R, γ_R) , $\gamma_R \in \Gamma_R$, $R \equiv N \setminus S$, such that $(x'_R, \gamma_R) \in A_-(R, V_{\gamma_S})$ there exists an **outcome** $\left((x'_S, x'_R), \gamma'\right)$, $\gamma' \in \Gamma(\gamma_S)$ such that $x' >_S x$. Consider a deviation by S from γ and the outcome (x,γ) and pick one (x'^*_R, γ^*_R) in $A_+(R, V_{\gamma_S})$. By the inductive assumption, this outcome is included in $A_-(R, V_{\gamma_S})$. Thus, there exists an **outcome** $\left((x'^*_S, x'^*_{N-S}), \gamma'\right)$, $\gamma' \in \Gamma(\gamma_S)$ such that $x'^* >_S x$. We find a coalition $S \subseteq N$, an outcome (x'^*_R, γ^*_R) from the residual game, and an outcome $\left((x'^*_S, x'^*_R), \gamma'\right)$ such that $x'^* >_S x$. Thus then (x,γ) is dominated (optimistic). This is a contradiction because $(x,\gamma) \in C_+(N,V)$. □

The Proposition 2 above defines a set interval of outcomes bounded below by $C_+(N,V)$ and above by $C_-(N,V)$.

Even though the generalized recursive core naturally extends the recursive core to the setting of overlapping coalitions, there are two fundamental problems with this notion. First, we must notice that the allocation $x_1 = \left((1,1,2)\right)$ from $z_1 = \left((1,1,2),\gamma_3\right)$, and $x_3 = \left((2,2,4)\right)$ from $z_3 = \left((2,2,4),\gamma_6\right)$ are core allocations but x_3 Pareto-dominates x_1. Why is this possible? The answer to that question is in the definition of the residual game. With our devotion to keep the tradition in cooperative game theory by defining the residual game on the complement of the set of the deviant coalition, we miss a subtle consistency requirement. In fact, a closer look at the definition of the residual game in the setting of partitions show three kinds of consistency. First, the set of players that perform a deviation forms a partition. Second, the remaining set of players react to this deviation forming a partition. Third, and the most important, the union of the two formed partition is a partition of the whole set of players. In our setting here, the benchmark definition of the residual game do not respect the third consistency requirement. Surely, the union of the two covers, from the deviation and from the residual game, form a cover. But not all covers can be partitioned in two disjoint covers. This is the case for example for each cover such that each coalition contains the same player. Thus in the Example 2, one can think of a deviation of all the payers from γ_3 (or γ_4) to γ_6 that is profitable since x_3 Pareto-dominates x_1. But this deviation is not possible with this benchmark definition of the residual game. In fact a deviation from γ_3 to γ_6 consists of the combination of a partial deviation by player k and a total deviation by the player j. This is a set deviation by $\{j,k\}$ but the reaction to that deviation is an action by player i. Therefore the cover γ_6 cannot form at equilibrium. In the following section, we propose another definition for the residual game, and we solve this inconsistency problem.

3 Overlapping coalition structure core

3.1 Overlapping residual game

In the previous section, we pay our willingness to keep the traditional game theoretical logic, to separate the residuals from the deviators, at the price of unsatisfactory results. We come to the evidence that we need to develop a new paradigm for the residual game in the setting of overlapping coalitions. In this section we introduce the *overlapping residual game*, with the novelty that it is not restricted to the complement of the set of deviators. Instead, the set of residual players in this section intersects the set of deviators via overlapping players. We ensure that this new paradigm meets the requirements of the three kinds of consistency. At this point, we need to introduce two extra definitions, *active* and *passive* players.

Definition 6 Suppose that a player set D, performs a deviation and forms a cover γ_D of D. The players in $N \setminus D$ are active players, and the overlapping players in D, O_D, are passive players.

The use of the terms passive and active is not fortuitous. In fact, passive players cannot form coalitions on their own, but active can. Passive players can only form coalitions with active ones. Therefore, they do not initiate an agreement but they can accept to sign new ones with active players without reneging on their previous coalitions.

Suppose that a player set $D \subseteq N$ performs a deviation forming a cover γ_D and let $R = (N \setminus D) \cup O_D$.

Definition 7 The overlapping residual game is the game (R, V_{γ_D}),
where V_{γ_D} is such that for all (S, λ_R), with $S \subseteq R$ and $\lambda_R \cup \gamma_D \in \Gamma$, $V(S, \lambda_R) \equiv V(S, \lambda_R \cup \gamma_D)$.
An embedded coalition (S, λ_R), from the overlapping residual game is such $(\lambda_R \cup \gamma_D)$ is a cover of N. We also denote the collection of all such embedded coalitions by Σ_R.

By the Definition 7 above, the set of residual players R (that play the overlapping residual game) intersects the set of deviators, and this is a new paradigm, essential to consistently define the recursive core in the setting of overlapping coalitions. Notice that we can obtain an embedded coalition (S, λ_R) such that λ_R is not a cover of R. This can happen if λ_R contains active players only. The only requirement is that $(\lambda_R \cup \gamma_D)$ is a cover of N.

Example 3 We consider in this example the same 3-person *CFG* (N,V) with $N=\{i,j,k\}$ as in the Example 2. If we consider for example a deviation from N by $\{j,k\}$ forming the cover $\{jk\}$, the residual game is played by the player set $R=\{i,k\}$ because i is an overlapping player in the *CFG* (N,V). In this case, player i is active and k is passive. The possible sets λ_R are $\{i\}$ and $\{ik\}$ but not $\{k\}$ because passive players cannot form coalitions on their own. Notice that $\lambda_R=\{i\}$ is not a cover of R but $\{i\}\cup\{jk\}$ is a cover of $N=\{i,j,k\}$.

Proposition 3 *The overlapping residual game is a CFG on its own, defined on the set of active and passive players.*

Proof.
Suppose that a player set D deviates forming γ_D and let $R=(N\setminus D)\cup O_D$. Let $\lambda_R\equiv\{S_1,S_2,...,S_k\}$ be a cover from the overlapping residual game.
First, if $\bigcup_{i=1}^k S_i = N\setminus D$, then there is no passive player in λ_R. Let $S_{k+1}\equiv O_D$. Thus, $\gamma_R\equiv\{S_1,S_2,...,S_k,S_{k+1}\}$ is a cover of R and let $\widetilde{V}_{\gamma_D}(S_{k+1},\gamma_R)\equiv 0$.
Second, if $N\setminus D\subsetneq\bigcup_{i=1}^k S_i$, then there are some passive players in λ_R. In this case let $S_{k+1}\equiv R\setminus\bigcup_{i=1}^k S_i$. Thus, $\gamma_R=\{S_1,S_2,...,S_k,S_{k+1}\}$ is a cover of R. Let $\widetilde{V}_{\gamma_D}(S_{k+1},\gamma_R)\equiv 0$.
For any other embedded coalition $(S,\gamma_R)\in\Sigma_R$, $\widetilde{V}_{\gamma_D}(S,\gamma_R)\equiv V_{\gamma_D}(S,\gamma_R)$.
Thus the game $(R,\widetilde{V}_{\gamma_D})$ is a cover function game defined on R. □

3.2 Dominance

With the introduction of the overlapping residual game, we need to redefine the notion of dominance.

Definition 8 **Optimistic** (respectively **Pessimistic**) **dominance**
An outcome (x,γ) is dominated via a coalition S forming a cover γ_S of S, if for **at least one** outcome (respectively **all** outcomes) $\big((x'_{N\setminus S},x'_{O_S}),\lambda_R\big)$ from the overlapping residual game, there exists an outcome $\big((x'_S,x'_{N\setminus S}),\gamma'\big)$, with $\gamma'\equiv\gamma_S\cup\lambda_R$, such that $x'>_S x$.

Notice carefully in the Definition 8 above that an outcome from the overlapping residual game is not sufficient to determine the overall payoff of overlapping players in S. We need to add also payoffs from γ_S. This subtle difference add to the complexity of the definition, and is actually the price to pay in order to consistently define the residual game in a setting where coalitions overlap.

3.3 Overlapping coalition structure core

Consider the game (N, V), of n players. In the following, we define the *overlapping coalition structure core*. It is defined recursively on the number of active players and it makes sense because no coalition can be formed in the residual game without active players.

(i) For a game with only one player, say $\{1\}$, the core is $C^*(\{1\}, V) = \big\{V(\{1\},\{1\}); \{1\}\big\}$.

(ii) Assume that the core $C^*(R, V)$ is defined for every game with k **active players**, $k = 1, 2, \ldots, n-1$. If the core is empty, define $A^*(R, V) \equiv \Omega(R, V)$, otherwise, define $A^*(R, V) \equiv C^*(R, V)$.

(iii) An outcome (x, γ) is dominated via a coalition S forming a cover γ_S of S, if for **at least one** outcome (respectively **all** outcomes) $\big((x'_{N\setminus S}, x'_{O_S}), \lambda_R\big)$ from $A^*(R, V_{\gamma_S})$, where $R \equiv (N \setminus S) \cup O_S$, there exists an outcome $\big((x'_S, x'_{N\setminus S}), \gamma'\big)$, with $\gamma' \in \Gamma(\gamma_S)$ such that $x' >_S x$.

(iv) The **optimistic** (respectively **pessimistic**) **overlapping coalition structure core** of an n-person game, $C^*_+(N, V)$ $\big($respectively $C^*_-(N, V)\big)$, is the set of non-dominated outcomes by optimistic (respectively pessimistic) dominance.

There is a subtle but big difference between the overlapping coalition structure core and the generalized recursive core. In fact at the point (iii) of the definition, we separate an outcome from the residual game in two parts, one for the overlapping players (x'_{O_S}) and the other for the remaining active players $(x'_{N\setminus S})$. This is, if a players set S deviates from a prescribed agreement forming γ_S, the total payoff awarded to the residual overlapping players who belong to λ_R, $R \equiv (N \setminus S) \cup O_S$, is the summation of their gain from λ_R and γ_S. Therefore, this payoff is not known until the final cover of N is completely formed. The following example will help fix ideas.

Example 4 We consider the same 3-person *CFG* as in the Example 2. The outcome $z_3 = \big((2, 2, 4), \gamma_6\big)$ is not dominated (pessimistic and optimistic) and therefore, belongs to the overlapping coalition structure cores ($C^*_+(N, V)$ and $C^*_-(N, V)$). In contrast to the Example 2, the outcomes $z_1 = \big((1, 1, 2), \gamma_3\big)$ and $z_2 = \big((1, 1, 2), \gamma_4\big)$ are dominated pessimistically.
From z_1, the player set $S = \{k, j\}$ will deviate to form the cover $\gamma_S = \{kj\}$. The residual game is played by $R = \{i, k\}$, forming $\lambda_R = \{ik\}$ and γ_3 forms. A similar pattern of deviation works for z_2 with the inversion of the role played by i and j.

As in the Proposition 1, it is easy to show that optimistic and pessimistic overlapping coalition structure cores coincide with these notions on partition function games. It is sufficient to notice that in the absence of overlapping players, there is no passive player and we obtain the same context as for Proposition 2.

Proposition 4 *Given a CFG* (N, V), $C^*_+(N, V) \subseteq C^*_-(N, V)$.

The proof is similar to the prof of the Proposition 2 with an induction on the number of active players. Based on this result, we can define a set interval for overlapping coalition structure core outcomes, bounded below by $C^*_+(N, v)$ and above by $C^*_-(N, v)$.
In the following, we compare the overlapping coalition structure core with the generalized recursive core.

Proposition 5 *Given a CFG* (N, V),

(i) $C^*_-(N, V) \subseteq C_-(N, V)$, *and*
(ii) $C^*_+(N, V) \subseteq C_+(N, V)$,

By the Proposition 5, the notion of overlapping coalition structure core defines a refinement of the generalized recursive core.

Proof.
We first prove (i) ($C^*_-(N, V) \subseteq C_-(N, V)$) by contraposition.
Consider an outcome (x, γ). If $(x, \gamma) \notin C_-(N, V)$, then it is dominated via a coalition S forming γ_S. This is, there exists coalition S and a payoff vector $(y_{N\setminus S})$ for players from $A_-(R, V_{\gamma_S})$ (who are active players) such that $y >_S x$. Now consider a payoff vector $(z_{N\setminus S})$ for active players from $A^*_-(R, V_{\gamma_S})$. Notice that $(z_{N\setminus S})$ comes from some $(z_R, \lambda_R) \in A^*_-(R, V_{\gamma_S})$ with $R \equiv (N \setminus S) \cup O_S$. Two cases are possible:
Case 1: No passive player in λ_R
In this case $\lambda_R \cap O_S = \emptyset$. Thus λ_R is a cover of $N \setminus S$ and $(z_{N\setminus S})$ is a payoff vector from $A_-(N \setminus S, V_{\gamma_S})$. Thus if S deviates forming γ_S, then the payoff vector $y = (z_S, z_{N\setminus S}, \gamma')$ is an outcome for the *CFG* (N, V) and $y >_S x$. Thus $(x, \gamma) \notin C^*_-(N, v)$.
Case 2: At least one overlapping player in λ_R
Decompose $\lambda_R = \{S_1, S_2, ..., S_k\}$ into two sets: $\gamma_1 \cup \gamma_2$ such that $\gamma_1 = \{S_i \in \lambda_R$ such that $S_i \cap O_S \neq \emptyset\}$ (containing all passive players linked in coalitions with some active ones), and $\gamma_2 = \{S_i \in \lambda_R$ such that $S_i \cap O_S = \emptyset\}$ (containing active players only). Note A, the union of all the sets in γ_1, and B the union of all the sets in γ_2. Consider the deviation by $S \cup A$ forming a cover $\gamma_S \cup \gamma_1$. Since $(z_R, \gamma_R) \in A^*_-(R, V_{\gamma_S})$, we have $(z_B, \gamma_2) \in A_-\big(N \setminus (S \cup A), V_{\gamma_{S\cup A}}\big)$. Furthermore the payoff vector $y = (z_{S\cup A}, z_B)$ and the cover $\gamma' = \gamma_S \cup \lambda_R = \gamma_S \cup \gamma_1 \cup \gamma_2$ are such that (y, γ') is an outcome for the *CFG* (N, V) and $y >_{S\cup A} x$. Hence, $(x, \gamma) \notin C^*_-(N, v)$.
In the following we prove (ii) ($C^*_+(N, V) \subseteq C_+(N, V)$) directly. Let $(x, \gamma) \in C^*_+(N, V)$ be an outcome for the *CFG* (N, V). Suppose that $(x, \gamma) \notin C_+(N, V)$. There exists a set of players S such that S deviates from γ forming γ_S and a payoff vector $y_{N\setminus S}$ and a cover $\gamma_{N\setminus S}$ from $A_+(R, V_{\gamma_S})$

such that $y >_S x$. Consider all the outcomes of the game (z, γ') of the *CFG* game (N, V) such that $z \geq_S y$. Considering a deviation by S, at least one of such $z = (z_S, z_{N\setminus S})$ comes from a deviation by S and the overlapping residual game played by $R = N \setminus S \cup O_S$, (if not $y, \gamma' \in A^*_+(R, V_{\gamma_S})$) and $z = (z_S, z_{N\setminus S})$ such that $z_{N\setminus S}$ comes from $A^*_+(R, V_{\gamma_S})$. Therefore, since $y >_S x$, we have $z >_S x$ and thus $(x, \gamma) \notin C^*_+(N, V)$ which is a contradiction. . □

Proposition 6 *The overlapping coalition structure cores are Pareto-efficient.*

Proof.
Suppose the overlapping coalition structure core $C^*(N, V)$ is non-empty and consider an outcome $(x, \gamma) \in C^*(N, V)$. Suppose to the contrary of the Proposition 6 that the outcome (x, γ) is Pareto-dominated by (x', γ'). This is $x' >_N x$. Consider the deviation by a player set S from γ that induces the formation of γ'. Thus obviously we have $x' >_S x$ because by assumption, $x' >_N x$. Therefore, (x, γ) is dominated, and it does not belong to the core. This is a contradiction. □

Notice that as in Koczy (2007), there is no need to distinguish between the pessimistic and the optimistic overlapping coalition structure core. Ultimately, the Proposition 6 reinforces us in the introduction of the new paradigm for the definition of the residual game.

4 Conclusion

This paper investigates possibilities to extend the notion of the core to the setting of overlapping coalitions. More precisely we develop two extensions of the recursive core, and consequently the coalition structure core. When we have developed the first extension in the logic of the duality "deviant coalition and complement residuals", as it is always done in the cooperative game theory, we did not expect the core to miss some consistency requirements. We solve this problem by proposing a new paradigm leading to a far better extension. Surely, the concept of the core is widely studied for partitions but very few solution concepts address the stability of overlapping coalitions. This paper should therefore be viewed as a step forward in the generalization of cooperative games. Since coalitions naturally overlap in most issues in economics where groups are involved in strategic interactions, the overlapping recursive core that we develop in this paper will help assess the stability of coalitions in such situations. However, we are aware that there is a long way to go concerning solution concepts for overlapping coalitions and we hope that our work will generate many others on the matter.

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