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On Income and Utility Generalised Transfer Principles in a Welfarist-Paretian Separable Framework

Marc DUBOIS
Stéphane MUSSARD

UNIVERSITÉ DE SHERBROOKE

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Marc Dubois*†
Lameta
Université de Montpellier

Stéphane Mussard‡
Chrome
Université de Nîmes

Abstract

The aim of the paper is to investigate the interplay between both transfer principles defined on incomes and utilities. For instance, a social planner who respects an income transfer principle of a given order is not necessarily incline to support the utility transfer principle of the corresponding order. Utility transfer principles are useful to characterize social planners' attitudes towards inequality but redistributive justice is often empirically assessed on income transfers, and as such it seems interesting to have an overview of different types of social planners who respect (or not) an income transfer principle. The main result of the paper is the determination of a generalised critical value that displays attitudes towards inequality which are necessary and sufficient for the respect of all income transfer principles up to any order. By using Bell's polynomial, we demonstrate that such a critical value alternates in sign according to the considered order. Our results give sufficient conditions to state stochastic dominance of income distributions as a factual moral statement.

Classification: D3; D6; H2. *Keywords*: Bell's polynomial, Inequality aversion, Prioritarianism, Transfer principles.

*Corresponding author. E-mail: dubois2@lameta.univ-montp1.fr.

†Lameta Université de Montpellier, Faculté d'Economie, Av. Raymond Dugrand, Site de Richter C.S. 79606, 34960 Montpellier Cedex 2, France. I wish to thank the members of the Grédi (Université de Sherbooke), in which some part of this work was written. I thank finally Saghar Saïdi for her suggestions. This article was presented in PET13 and ECINEQ 2013 conferences.

‡Chrome Université de Nîmes - e-mail: smussard@adm.usherbrooke.ca, Associate researcher at Grédi, Université de Sherbrooke, and CEPS Luxembourg. We greatly thank Mickaël Beaud for helping us to establish the concern of the paper and to prove the lemma.

1 Introduction

Whenever income inequality is considered as a social cost, the Pigou-Dalton income transfer principle states that a richer-to-poorer income transfer improves social welfare. This principle is the corner stone of the inequality measurement literature, in which inequality measures are issued from an additively separable social welfare function (SWF). The most commonly employed SWF is the utilitarian one, which simply sums individual utilities. As suggested by Adler (2012), the Pigou-Dalton transfer principle could be defined over utilities rather than incomes. Accordingly, the traditional inequality aversion would be the result of the improvement of social welfare inherent to a *progressive utility transfer* i.e. a transfer of utility between a well-off person to a worse-off one[1]. As a consequence, the inequality aversion is the main property of prioritarian SWFs.[2] Therefore, an SWF that fulfils the Pigou-Dalton utility transfer principle is inequality averse or, in Nagel's terms, it is "egalitarian in herself". Furthermore, Nagel (1995, p. 65) points out:

> "Even if impartiality were not in this sense egalitarian in itself, it would be egalitarian in its distributive consequences because of the familiar fact of diminishing marginal utility".

This quotation may be understood as follows: the Pigou-Dalton utility transfer principle is sufficient but not necessary for income redistribution to be desirable "because of the familiar fact of the diminishing marginal utility" of income. This is the concern of the present paper.

We consider every individual of a given population in an indealized position as a social planner seeking to rank some income distributions by means of a social welfare function that sums the transformed utilities. All social planners first assume a strictly increasing and concave utility function aiming at stating individual utilities. Second, they transform utilities according to their attitudes towards inequality. These attitudes are embodied by the respect (disrespect) of utility transfer principle at any given order. To clarify, the second-order utility (or income) transfer principle is the Pigou-Dalton utility (or income) transfer principle and the third-order utility (or income) transfer principle is Kolm's (1976) utility (or income) *diminishing transfers principle*. The latter states that a progressive transfer is more valuable, the worse-off (poorer) the recipient is. Higher-order utility transfer principles rely on Fishburn and Willig's (1984) concept of transfer function of utility, initially defined for generalised income transfer principle.

[1] As for the Pigou-Dalton transfer principle applied to incomes, the recipient's utility improvement has the same magnitude as the donor's utility loss.

[2] Precisely, Adler (2012) defines an SWF as Prioritarian if it is characterized by Strong Pareto, Separability and the Pigou-Dalton utility transfer principle.

The aim of the paper is to investigate the interplay between both utility and income transfer principles. Precisely, as pointed out by Nagel, a social planner who respects the second-order *income* transfer principle is not necessarily incline to support the second-order *utility* transfer principle. In a more general fashion, a social planner who respects a given order of income transfer principle is not necessarily incline to support the corresponding order of utility transfer principle. Also, it turns out that some incompatibilities occur, for instance, a social planner who respects both second-order and third-order income transfer principles cannot disrespect both second-order and third-order utility transfer principles. Redistributive justice is often empirically assessed on income transfers, and as such, it seems interesting to have an overview of the different types of social planners who respect (or not) an income transfer principle whereas utility transfer principles are disrespected (or not).

Precisely, the S-concavity of the SWF is necessary and sufficient to fulfil the Pigou-Dalton income transfer principle, but it is not a sufficient condition to fulfil the corresponding utility transfer principle. Actually, a social planner respects the Pigou-Dalton income transfer principle if, and only if, the value of the second derivative of the transformation function does not exceed a critical value. We demonstrate that this critical value must be positive. Thus, even if the transformation function is convex, reflecting inequality loving, then a social planner may however respect the Pigou-Dalton income transfer principle. Going one step further, an inequality averse social planner assuming a strictly increasing utility function with its three first successive derivatives that alternate in sign respects the income diminishing transfers principle if, and only if, the value of the third derivative of the transformation function is at least as great as a critical value. We demonstrate that this critical value must be negative. In this case, even if the third derivative of the transformation function is negative, reflecting preference for progressive utility transfers involving a better-off recipient (rather than involving a worse-off recipient), then a social planner may however respect the income diminishing transfers principle. The main result of the paper is the determination of a generalised critical value that displays attitudes towards inequality, which are necessary and sufficient for the respect of all income transfer principle up to any order. Following Fishburn and Willig (1984), the ranking of income distributions generated by SWFs that fulfil all the income transfer principles up to a given order is equivalent to that of stochastic dominance of income distributions at the corresponding order.

As Adler (2012) points out, in our framework there is room for various meta-ethical positions. One of them asserts that if all considered social planners' rankings of income distributions are convergent, then this ranking is a moral fact. We try to enlarge up to necessary the set of attitudes towards inequality that respect all income transfer principles up to any order, doing so we give sufficient conditions for stating stochastic dominance of income distributions as moral facts.

The paper is organised as follows. In section 2 we present the setup and the main results of the paper. Section 3 closes the paper by providing some discussions about the results and a table summarising them.

2 Setup and results

Income is denoted by $y \in \Omega := [0, y_{\max}]$ where $y_{\max}$ is the maximum conceivable one. Let $f(y)$ be the probability density function (p.d.f.) at income y such that $f(y_{\max}) > 0$. The set $\mathscr{F}$ of relevant p.d.f. is the set of real-valued functions from Ω into $[0, 1]$ such that, for each f in $\mathscr{F}$, $f(y) > 0$ for at most a finite number of y, with $\int_0^{y_{\max}} f(y)dy = 1$.

The additively separable SWF is a tool employed by a social planner in order to morally rank income distributions:[3]

$$W(f) = \int_0^{y_{\max}} g \circ u(y)\ f(y)\ dy. \tag{1}$$

Let $g \circ u(y)$ be the moral value, determined by a social planner, of the utility level generated by income y, such that $u(y) \geqslant 0$ for all $y \in \Omega$. The function g formalises a social planner's attitudes towards inequality. It is increasing over utility by assumption, *i.e.* $g^{(1)}(u(y)) > 0$ for all $y \in \Omega$, and it belongs to the set of s-time differentiable functions $\mathscr{C}^s$, such that $s \in \mathbb{N} := \{1, 2, \ldots\}$. Let us denote by $g^{(s)}(u)$ the s-order derivative of g with respect to u. In the remainder, utility functions are supposed to be strictly increasing, $u^{(1)}(y) > 0$ for all $y \in \Omega$, and they also belong to $\mathscr{C}^s$.

We define Transfer functions $T(\cdot)$ *à la* Fishburn and Willig (1984) in order to deal with a wide range of redistributive principles based either on utility or on income. On the one hand, the utility transfer of order $s = 1$, $T^1(\alpha, u(y), \delta)$ postulates that the proportion $\alpha \in (0, f(y)]$ of the population moves from utility $u(y)$ to utility $u(y) + \delta$ such that $\delta > 0$. For all $f \in \mathscr{F}$ and for all $y \in \Omega$, a p.d.f. $h(y)$ is obtained from $f(y)$ by a utility transfer of order 1 whenever:

$$h(y) = \begin{cases} f(y) - \alpha & \text{at point } u(y)\ , \\ f(y) + \alpha & \text{at point } u(y) + \delta\ , \\ f(y) & \text{elsewhere.} \end{cases} \tag{2}$$

Utility transfers of order 1 do not preserve the mean utility generated by f. For higher orders, utility mean-preserving transfers of order $s + 1$ are deduced recursively.

[3]A moral ranking relies on a moral view. Here we consider that a moral view is welfarist-paretian and respects *separability* and *continuity*. *Separability with respect to unconcerned individuals* requires the ranking of two income distributions to depend only on the well-being of the concerned individuals; *i.e.* those who are strictly better-off or worse-off in one distribution than in the other (this axiom is called as such in d'Aspremont and Gevers (1977)). Following Adler (2012, p. 311), continuity demands that "'a small change in individual utilities does not produce a large change in the moral ranking"' of income distributions. This set of axioms implies the relation "'morally at least as good as"' to be represented by an additively separable SWF.

Definition 2.1. Generalised Utility Transfer of order $s+1$**.** *For all* $f \in \mathscr{F}$ *and for all* $y \in \Omega$*, a utility mean-preserving transfer of order* $s+1$ *is given by:*

$$T^{s+1}(\alpha, u(y), \delta) := T^s(\alpha, u(y), \delta) - T^s(\alpha, u(y)+\delta, \delta), \;\; s \in \mathbb{N}_+, \tag{3}$$

such that, for $r \in \{1, \cdots, s+1\}$ *being even,*[4]

$$\binom{s+1}{r} \alpha \in (0, f(y)] \text{ at point } u(y) + r\delta, \text{ and } \delta > 0.$$

On the other hand, based on the same kind of p.d.f. transformation as (2), the income transfer of order $s = 1$, $T^1(\alpha, y, \delta)$ postulates that the proportion $\alpha \in (0, f(y)]$ of the population moves from income y to income $y+\delta$ such that $\delta > 0$. For higher orders, mean-preserving income transfers of income of order $s+1$ are deduced recursively.

Definition 2.2. Generalised Income Transfer of order $s+1$**.** *For all* $f \in \mathscr{F}$ *and for all* $y \in \Omega$*, an income mean-preserving transfer of order* $s+1$ *is given by:*

$$T^{s+1}(\alpha, y, \delta) := T^s(\alpha, y, \delta) - T^s(\alpha, y+\delta, \delta), \;\; s \in \mathbb{N}_+, \tag{4}$$

such that, for $r \in \{1, \cdots, s+1\}$ *being even,*[5]

$$\binom{s+1}{r} \alpha \in (0, f(y + r\delta)], \;\; \delta > 0.$$

Fishburn and Willig (1984, p. 323) define the *transfer principle Ts* (*i.e.* of order s) as a "conjunction of a natural extension of the [Pigou-Dalton] principle". In this sense, the principle of order s should incorporate all transfer principles of lower-order than s. In our framework, the generalised transfer principle of order s does not embody lower-order ones.

Definition 2.3. Generalised Utility and Income Transfer Principle of order $s+1$**.** *For all* $f \in \mathscr{F}$*, for all* $y \in \Omega$ *and* $s \in \mathbb{N}$*, we have the following implications for utility and income transfers respectively:*

$$h = f + T^{s+1}(\alpha, u(y), \delta) \implies W(h) \geqslant W(f) \tag{UTP^{s+1}}$$

$$h = f + T^{s+1}(\alpha, y, \delta) \implies W(h) \geqslant W(f). \tag{ITP^{s+1}}$$

[4]This condition recalls that the proportion of population who moves from a utility level should be at most as high as the proportion of population at this utility level. Moreover it demands that no proportion moves from a utility level generated by a higher income than $y_{\max}$.

[5]This condition implies that $y \in [0; y_{\max} - s\delta]$ whenever s is even, and $y \in [0; y_{\max} - (s-1)\delta]$ whenever s is odd. This necessary condition for income transfers does not appear in Fishburn and Willig (1984) because they adopt an unbounded income domain whereas here $y \in [0; y_{\max}]$.

This generalisation enables particular utility transfers to be captured. For instance, the transfer function $T^2(\alpha, u(y), \delta)$ is a particular case of utility progressive transfer. It is a utility transfer of amount δ from a better-off $(u(y) + 2\delta)$ proportion α of the population to a worst-off $(u(y))$ proportion α. Hence, a part of the population moves from the utility level $u(y)$ to $u(y) + \delta$ and another part moves from the utility level $u(y) + 2\delta$ to $u(y) + \delta$. Let us remark that T^2 equalizes the utility levels of people involved in the transfer, and as such, it is an *equalizing utility transfer*.

On the one hand, in the UTP^2 definition, it clearly appears that a transfer among two equal proportions of population improves the social welfare *only if* the transfer equalizes the utility levels of both involved proportions. On the other hand, the Pigou-Dalton transfer principle of utility requires that a transfer from a better-off individual to a worse-off one that reduces the gap between both utility levels improves the social welfare.[6] Clearly, the Pigou-Dalton utility transfer principle is stronger than UTP^2.

The transfer function $T^3(\alpha, u(y), \delta)$ encompasses (i) an equalizing utility transfer and (ii) a *disequalizing utility transfer* involving a proportion of better-off recipients than in (i). The part (i) is a transfer $T^2(\alpha, u(y), \delta)$. The part (ii) is a transfer $-T^2(\alpha, u(y)+\delta, \delta)$: it is a utility transfer of amount δ from a proportion of the population α with utility level $u(y) + 2\delta$, to a proportion α with the same utility level. UTP^3 involves the T^3 transfer. It is a particular case of the utility diminishing transfers principle.[7] Fishburn and Willig (1984) put a lot of structure in the transfers T^{s+1} in order to provide a recursive and general formulation. We follow the same idea.

The SWF (1) satisfies UTP^2 if, and only if, $g^{(2)}(u(y)) \leqslant 0$ for all $y \in \Omega$. The concavity of g characterizes inequality aversion. Following Adler (2012), an SWF (1) that displays inequality aversion is considered as prioritarian. The SWF (1) satisfies UTP^3 if, and only if, $g^{(3)}(u(y)) \geqslant 0$ for all $y \in \Omega$.[8] If (1) fulfils UTP^2 and UTP^3, then it exhibits more inequality aversion for the worse-off than for the better-off proportions of the population.

Fishburn and Willig's (1984) generalised transfer functions T^{s+1}, are associated with the following set:

$$\Gamma^{s+1} := \left\{ f \in \mathscr{C}^{s+1} \;\middle|\; (-1)^{s+1} f^{(s+1)}(x) := (-1)^{s+1} \frac{\partial f^{(s)}(x)}{\partial x} \leqslant 0, \quad \forall x \in \mathbb{R}_+ \right\},$$

[6] There are various understandings about the gap reduction. See Thon and Wallace (2004) for a discussion in which the transfer principle implies symmetry. See Fields and Fei (1978) for a presentation of a rank-preserving transfer principle. It is worth mentioning that both references discuss income transfer principles.

[7] One could object that the equalizing utility transfer and the disequalizing one involve different utility gaps between the respective donors and recipients. But let remark that $W(h) > W(f)$ with $h = f + T^3(\alpha, u(y), \delta)$ is equivalent to $W(f + T^2(\alpha, u(y), \delta)) > W(f + T^2(\alpha, u(y) + \delta, \delta))$ in which case both equalizing utility transfers involve the same utility gaps between the respective donors and recipients.

[8] Let us remark that the concavity of g is not necessary to respect UTP^3. This statement is just a corollary of Theorem 1 in Chateauneuf *et al.* (2002).

i.e. the class of real-valued functions, which share the property of having their $s+1$th derivative non-positive (non-negative) at an even (odd) order $s+1$, where $s \in \mathbb{N}_+$. Hence,

$$\Gamma^{\rightarrow s+1} := \left\{ f \in \mathscr{C}^{s+1} \,\middle|\, f \in \Gamma^{\ell}, \quad \forall \ell = 2, \ldots, s \right\},$$

is the set of all $s+1$-time differentiable functions for which their first $s+1$ successive derivatives alternate in sign. Formally, $\Gamma^{\rightarrow s+1} = \Gamma^2 \cap \cdots \cap \Gamma^{s+1}$.

Theorem 2.1. *For any given $s \in \mathbb{N}$, the two following statements are equivalent:*

(i) *W satisfies all utility transfer principles up to the order $s+1$.*

(ii) $g \in \Gamma^{\rightarrow s+1}$.

Moreover the two following statements are equivalent:

(iii) *W satisfies all income transfer principles up to the order $s+1$.*

(iv) $g \circ u \in \Gamma^{\rightarrow s+1}$.

Proof. This is a direct application of Fishburn and Willig's (1984) result (Theorem 1). □

Theorem 2.1 yields normative interpretations about the respect of all utiliy transfer principles up to any order. The SWF (1) fulfils the second-order UTP^2 if, and only if it exhibits inequality aversion (*i.e.* g is convave). Let the condition (a) $g \in \Gamma^{\rightarrow s}$ become (b) $g \in \Gamma^{\rightarrow s+1}$; this is equivalent to assert that the SWF (1) fulfiling (b) places more emphasis on inequality for the worse-off proportions than the SWF (1) fulfiling (a). For instance, the SWF (1) that fulfils both UTP^2 and UTP ^{3} displays more inequality aversion for the worse-off than for the better-off part of the distribution whereas the SWF (1) that fulfils only UTP^2 displays at least as inequality aversion for the better-off as for the worse-off part of the distribution. As an extreme case, the SWF (1) with $g \in \Gamma^{\rightarrow \infty}$ displays an absolute inequality aversion for the worst-off proportion of the population. It fulfils all the utility transfer principles and represents the leximin criterion.

Let us now turn to the second-order Income Transfer Principle (ITP^2): if a transfer of a positive amount of income equalizes the income levels of a poorer and a richer proportion of the population, then this transfer improves the social welfare. This transfer principle is weaker than the Pigou-Dalton income transfer principle. Obviously, if $u(\cdot) \in \Gamma^{\rightarrow 2}$ and $g(\cdot) \in \Gamma^{\rightarrow 2}$, then the SWF (1) fulfils ITP^2. In our framework, the respect of ITP^{s+1} depends on two sufficient conditions. The first is related to the shape of u, the second is based on a social planner's attitudes towards inequality, *i.e.* the shape of g.

Lemma 2.1. *The following statement is true for all $s \in \mathbb{N}$:*

$[\mathbf{H}^{s+1}]$: $u \in \Gamma^{\rightarrow s+1}$ *and* $g \in \Gamma^{\rightarrow s+1}$ *together imply* $g \circ u \in \Gamma^{\rightarrow s+1}$.

Proof. See the Appendix. □

If we consider an inequality neutral social planner such that she determines the moral value due to income y as $u(y)$ rather than $g \circ u(y)$, then the condition $u \in \Gamma^{\to s+1}$ is necessary and sufficient to respect all income transfer principles up to order $s+1$. Lemma 2.1 shows that the conditions for the fulfilment of ITP^{s+1} are only sufficient ones. Necessary and sufficient conditions may be derived from Lemma 2.1 and Theorem 2.1 in the following way.

Theorem 2.2. *Let $u \in \Gamma^{\to s+1}$ and $g \in \{\Gamma^{\to s} \cap \mathscr{C}^{s+1}\}$ such that $s \in \mathbb{N}$. Then, the two following statements are equivalent:*

(i) *SW satisfies all income transfer principles up to the order $s+1$.*

(ii) *$(-1)^{s+1} g^{(s+1)} \circ u \leqslant (-1)^{s+1} g^{\star(s+1)}$ such that $(-1)^{s+1} g^{\star(s+1)} \geqslant 0$, with:*

$$g^{\star(s+1)} = -\frac{\sum_{k=1}^{s} (g^{(k)} \circ u) \cdot B_{s+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(s-k+2)}\right)}{\left[u^{(1)}\right]^{i+1}},$$

where $B_{s+1,k}(\cdot)$ is Bell's polynomial of order $s+1, k$.

Proof. See the Appendix. □

3 Concluding Remarks

Remark 1. The line of though of Theorem 2.2 can be presented in 3 steps: (i) an SWF (1) fulfils the income transfer principles up to order $s+1$ if, and only if, $g \circ u \in \Gamma^{\to s+1}$. It is the cornerstone of the theorem and it is a simple corollary of Theorem 1 of Fishburn and Willig (1984). For instance, the SWF (1) fulfils ITP^2 if, and only if, the composition function $g \circ u$ is concave. (ii) We aim at determining what the necessary and sufficient condition is, related to the shape of $g(\cdot)$ to fulfil all ITPs up to the order $s+1$. Formally, $(-1)^{s+1} g^{(s+1)} \circ u \leqslant (-1)^{s+1} g^{\star(s+1)}$ if, and only if, (1) fulfils all ITPs up to the order $s+1$. For example, the SWF (1) fulfils ITP^2 if, and only if, $g^{(2)} \leqslant g^{\star(2)}$. (iii) If one makes assumptions on the shape of $u(\cdot)$, it becomes possible to determine the sign of the boundary $g^{\star(s+1)}$. Hence, this boundary becomes interpretable. In Theorem 2.2, conditions $u \in \Gamma^{\to s+1}$ and $g \in \{\Gamma^{\to s} \cap \mathscr{C}^{s+1}\}$ imply the second part of the theorem: $(-1)^{s+1} g^{\star(s+1)} \geqslant 0$. To illustrate this, setting $u \in \Gamma^{\to 2}$ yields $g^{\star(2)} \geqslant 0$. In other words, if one assumes that individuals have increasing but marginally decreasing utility over income, then the concavity of $g(\cdot)$ is sufficient but not necessary to fulfill ITP^2.[9] The SWF (1) satisfying ITP^2 can include a g function being either concave or convex on the domain, so long as the value of its second derivative is not higher than a critical value $g^{\star(2)}$. Given that $g(\cdot)$ represents attitude towards inequality,

[9] A similar result could hold if we adopt a stronger version of ITP^2 as the Pigou-Dalton transfers Principle of income, for instance.

the respect of ITP^2 does not necessarily imply inequality aversion (that is concavity of $g(\cdot)$). We retrieve Nagel's assertion presented in the introduction. Prioritarianism is sufficient but not necessary for the respect of ITP^2. Inequality-loving is compatible with ITP^2 and ITP^3 (Table 1, Line 2) but we do not investigate these cases in the paper.

Table 1. Value Judgments and Income Transfer Principles.

UTP^2	UTP^3	ITP^2 Respect, ITP^3 Respect [ITP^2+ITP^3]	ITP^2 Respect, ITP^3 Disrespect	ITP^2 Disrespect, ITP^3 Respect [ITP^3]	ITP^2 Disrespect, ITP^3 Disrespect
Disrespect	Disrespect	Impossible	$g^{(2)} \geqslant 0^*$ but $g^{(2)} \leqslant g^{\star(2)}$ $g^{(3)} \leqslant 0$ and $g^{(3)} \leqslant g^{\star(3)*}$	$g^{(2)} \geqslant 0^*$ and $g^{(2)} \geqslant g^{\star(2)}$ $g^{(3)} \leqslant 0$ but $g^{(3)} \geqslant g^{\star(3)}$	$g^{(2)} \geqslant 0^*$ and $g^{(2)} \geqslant g^{\star(2)}$ $g^{(3)} \leqslant 0$ and $g^{(3)} \leqslant g^{\star(3)*}$
Disrespect	Respect	$g^{(2)} \geqslant 0$ but $g^{(2)} \leqslant g^{\star(2)}$ $g^{(3)} \geqslant 0$	Impossible	$g^{(2)} \geqslant 0^*$ and $g^{(2)} \geqslant g^{\star(2)*}$ $g^{(3)} \geqslant 0$	Impossible
Respect	Disrespect	$g^{(2)} \leqslant 0$ $g^{(3)} \leqslant 0$ but $g^{(3)} \geqslant g^{\star(3)}$ Theorem 2.2	$g^{(2)} \leqslant 0$ $g^{(3)} \leqslant 0$ and $g^{(3)} \leqslant g^{\star(3)*}$	Impossible	Impossible
Respect	Respect	$g^{(2)} \leqslant 0$ $g^{(3)} \geqslant 0$ Theorem 2.2	Impossible	Impossible	Impossible
Dominance		TSD (column 1); SSD (columns 1–2)			

*for at least some defined $u(y)$.

Remark 2. Aaberge (2009) proposes various attitudes to inequality well adapted to a rank-dependent framework. Taking recourse to those various attitudes to inequality in the present context, we get two types of social planners. Downward social planners who have more sensitivity for utility transfers the worse-off the recipient is ($g^{(3)} \geqslant 0$: Table 1, Lines 2 and 4). Upward social planners who have more sensitivity for utility transfers the better-off the recipient is ($g^{(3)} \leqslant 0$: Table 1, Lines 1 and 3). Theorem 2.2 shows that the respect of ITP^3 does not systematically imply Downward inequality aversion. The social planner may be Downward inequality-loving (Table 1: Line 2, Column 3). Table 1 also provides normative interpretations consistent with second-order stochastic dominance (SSD) and third-order stochastic dominance (TSD).

Remark 3. Following step (ii) of the line of though of Theorem 2.2 presented above: we aim at determining the necessary and sufficient condition on the shape of $g(\cdot)$ to fulfil all the ITPs up to the order $s+1$. However another step could be (ii'): we aim at determining the necessary and sufficient condition on the shape of $u(\cdot)$ to fulfil all the ITPs up to the order $s+1$. The choice of (ii) can be defended as follows. The function g represents a social planner's *value judgment* whereas u is understood as social planners' hypothesis they make on that they could observe in an idealized position: a *factual judgment*. Railton (1986) gives an interesting definition of the difference between *value judgment* and *factual judgment*. Let us consider two rational individuals. One of them asserts that $\mathcal{S}_1$ is better than $\mathcal{S}_2$, the other one asserts that $\mathcal{S}_2$ is better than $\mathcal{S}_1$. If idealized conditions allow for demonstrate logically that $\mathcal{S}_1$ is better than $\mathcal{S}_2$, then both individuals will converge to this assertion in an idealized position. In this case, assertions are factual judgments. However if one demonstrates under idealized conditions by all possible ways that $\mathcal{S}_1$ is better than $\mathcal{S}_2$, whereas one individual is still convinced that $\mathcal{S}_2$ is better than $\mathcal{S}_1$, then the assertions are value judgments.

The present welfarist-paretian framework in which Adler (2012) incorporates the prioritarian moral view is well-adapted to two meta-ethical views: cognitivism and noncognitivism. From noncognitivism, Railton's (1986) assertion makes no sense and all components of the SWF (1) are value judgments. In such a sense, our results are of interest but the paper could be completed by some results from step (ii'). From cognitivism, particularly Smith's (1994) view, each individual of the population in an idealized position makes a moral ranking of two income distributions by making value judgments (represented by g here). If the verdicts of all social planners are convergent, then the ranking of the distributions is a moral fact. We try to "enlarge" up to necessary the set of g functions that respect all the income transfer principles up to any order. The aim is to possibly present a set that embodies all social planners' attitudes towards inequality, in which case a moral ranking can be considered as a moral fact. For instance, $\int_0^{y_{\max}} g \circ u(y)\ f(y)\ dy \geqslant \int_0^{y_{\max}} g \circ u(y)\ h(y)\ dy$ for all $g \in \{g \in \mathscr{C}^2 : g \leqslant g^{\star(2)}\}$ if, and only if, the income distribution related to f is morally at least as good as the one associated to h. If $g \in \mathscr{C}^2 : g \leqslant g^{\star(2)}$ embodies all social planners' attitudes, then the above statement can be considered as a moral fact.

The ranking of income distributions generated by SWFs that fulfil all the income transfer principles up to a given order is equivalent to that of (weak) stochastic dominance of the considered distributions at the corresponding order. For instance, the previous example is equivalent to: f weakly stochastic dominates h at order 2. It turns out that if $g \in \mathscr{C}^2 : g \leqslant g^{\star(2)}$ embodies all social planners' attitudes, then the second-order stochastic dominance can be viewed as a moral factual statement. This idea can be generalised for any order of dominance.

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Appendix

For the sake of clarity, we do not make appear the argument of the u function in the proofs. Obviously it does not change anything to the results.

Proof. Lemma 2.1:
[⇒] We proceed by mathematical induction, *i.e.*, we first prove that the statement $\mathbf{H}^{s+1}$ is true for $s = 1$ and then we prove that if $\mathbf{H}^{s+1}$ is assumed to be true for any positive integer

s, then so is $\mathbf{H}^{s+2}$. Let $w := g \circ u$. It is apparent from $w^{(2)} = g^{(1)} u^{(2)} + g^{(2)} \left[u^{(1)}\right]^2$ that $\mathbf{H}^2$ is true. Let us now assume that the statement $\mathbf{H}^{s+1}$ is true in order to show that $\mathbf{H}^{s+2}$ holds. We proceed by remarking that:

$$(g \circ u)^{(s+2)} = \left[-g^{(1)} \circ u \cdot \left(-u^{(1)}\right)\right]^{(s+1)} \ .$$

We then have:

$$(-1)^{s+2} (g \circ u)^{(s+2)} = (-1)^{s+2} \left[-g^{(1)} \circ u \cdot \left(-u^{(1)}\right)\right]^{(s+1)} \ .$$

Remembering Leibniz' relation for the $(s+1)$-st derivative of the product of two functions $h \cdot f$:

$$(h \cdot f)^{(s+1)} = \sum_{k=0}^{s+1} \left(\begin{array}{c} s+1 \\ k \end{array} \right) \left[h^{(k)} \cdot f^{(s-k+1)}\right] \ ,$$

then:

$$(-1)^{s+2} (g \circ u)^{(s+2)} = (-1)^{s+2} \sum_{k=0}^{s+1} \left(\begin{array}{c} s+1 \\ k \end{array} \right) \left[(-g^{(1)} \circ u)^{(k)} \cdot (-u^{(1)})^{(s-k+1)}\right] \ .$$

Rearranging the terms yields:

$$(-1)^{s+2} (g \circ u)^{(s+2)} = (-1) \sum_{k=0}^{s+1} \left(\begin{array}{c} s+1 \\ k \end{array} \right) \left[(-1)^k \ (-g^{(1)} \circ u)^{(k)} \cdot (-1)^{s-k+1} \ (-u^{(1)})^{(s-k+1)}\right] \ . \tag{A0}$$

Now we want to prove the relation $\mathbf{H}^{s+2}$. Then we assume that $-g^{(1)} \in \Gamma^{\rightarrow s+1}$ (*i.e.* $g \in \Gamma^{\rightarrow s+2}$) and $u \in \Gamma^{\rightarrow s+2}$. The induction hypothesis $\mathbf{H}^{s+1}$ is supposed to be true, then for two functions f and h:

$$\text{if } f \in \Gamma^{\rightarrow s+1} \ \text{ and } \ h \in \Gamma^{\rightarrow s+1} \ , \text{ then } \ f \circ h \in \Gamma^{\rightarrow s+1}.$$

Let $f := -g^{(1)}$ and $h := u$ (actually if $u \in \Gamma^{\rightarrow s+2}$ then $u \in \Gamma^{\rightarrow s+1}$), then we obtain that $(-1)^k \ (-g^{(1)} \circ u)^{(k)} \leqslant 0$, for all $k = 0, \ldots, s+1$. If $u \in \Gamma^{\rightarrow s+2}$, then $(-1)^{s-k+2} u^{(s-k+2)} \leqslant 0$ for all $k = 0, \ldots, s+1$. Since Eq.(A0) can be expressed as,

$$(-1)^{s+2} (g \circ u)^{(s+2)} = (-1) \sum_{k=0}^{s+1} \left(\begin{array}{c} s+1 \\ k \end{array} \right) \left[(-1)^k \ (-g^{(1)} \circ u)^{(k)} \cdot (-1)^{s-k+2} \ u^{(s-k+2)}\right] \ ,$$

consequently, $(-1)^{s+2} (g \circ u)^{(s+2)} \leqslant 0$. As the statement $\mathbf{H}^{s+1}$ has been shown to be true for $s = 1$ and that the statement $\mathbf{H}^{s+2}$ has been proven to be true when $\mathbf{H}^{s+1}$ is invoked, then $\mathbf{H}^{s+1}$ is true.

[$\Leftarrow\!\!\!\!\!/$] It is left for the reader. □

Proof. Theorem 2.2:
From Fishburn and Willig (1984), $SW(F) = \int w(y)dF(y)$ respects ITP^{i+1} if, and only if, $(-1)^{i+1}\ w^{(i+1)}(y) \leqslant 0$ for all $i \in \{1,\dots,s\}$. Setting $w = g \circ u$, we get from Lemma 2.1 that $(-1)^i(g\circ u)^{(i)} \leqslant 0$ for all $i \in \{1,\dots,s\}$ since $g \in \Gamma^{\to s}$ and $u \in \Gamma^{\to s}$ (actually $u \in \Gamma^{\to s+1}$). It remains to find the conditions such that ITP^{s+1} is respected, *i.e.*, $(-1)^{i+1}(g\circ u)^{(i+1)} \leqslant 0$ for $i = s$. We start the demonstration with Faà di Bruno's formula:

$$(g\circ u)^{(i+1)} = \sum_{k=1}^{i+1}(g^{(k)}\circ u)\cdot B_{i+1,k}\left(u^{(1)},u^{(2)},\dots,u^{(i-k+2)}\right),$$

where Bell's polynomial $B_{i+1,k}(\cdot)$ is given by,

$$B_{i+1,k}\left(u^{(1)},u^{(2)},\dots,u^{(i-k+2)}\right) = \\ \sum \frac{(i+1)!}{p_1!p_2!\cdots p_{i-k+2}!}\left(\frac{u^{(1)}}{1!}\right)^{p_1}\left(\frac{u^{(2)}}{2!}\right)^{p_2}\cdots\left(\frac{u^{(i-k+2)}}{(i-k+2)!}\right)^{p_{i-k+2}}. \tag{A1}$$

The summation inside Bell's polynomial is taken over all possible sequences of integers $p_1,\dots,p_{i-k+2}$ such that $p_1+p_2+\cdots = k$ and $1p_1+2p_2+\cdots = i+1$. Since $(g\circ u)^{(i+1)} = (g^{(i+1)}\circ u)\cdot B_{i+1,i+1}(u^{(1)}) + \sum_{k=1}^{i}(g^{(k)}\circ u)\cdot B_{i+1,k}\left(u^{(1)},u^{(2)},\dots,u^{(i-k+2)}\right)$, then ITP^{i+1} is respected if, and only if,

$$(-1)^{i+1}(g\circ u)^{(i+1)} \leqslant 0 \\ \iff (-1)^{i+1}g^{(i+1)}\circ u \leqslant (-1)^{i+1}\left[-\frac{\sum_{k=1}^{i}(g^{(k)}\circ u)\cdot B_{i+1,k}\left(u^{(1)},u^{(2)},\dots,u^{(i-k+2)}\right)}{B_{i+1,i+1}(u^{(1)})}\right] =: (-1)^{i+1}g^{\star(i+1)}. \tag{A2}$$

As $B_{i+1,i+1}(u^{(1)}) = [u^{(1)}]^{i+1}$, thus:

$$g^{\star(i+1)} = -\frac{\sum_{k=1}^{i}(g^{(k)}\circ u)\cdot B_{i+1,k}\left(u^{(1)},u^{(2)},\dots,u^{(i-k+2)}\right)}{[u^{(1)}]^{i+1}}.$$

In order to derive the sign of the boundary $g^{\star(i+1)}$, we have to find the sign of $B_{i+1,k}(\cdot)$. Accordingly, we decompose the products of the functions $u^{(\cdot)}$ in $B_{i+1,k}(\cdot)$ in Eq.(A1) thanks to the following set

$$\Omega(k) := \left\{\underbrace{u^{(1)},\dots,u^{(1)}}_{p_1},\ \underbrace{u^{(2)},\dots,u^{(2)}}_{p_2},\dots,\ \underbrace{u^{(i-k+2)},\dots,u^{(i-k+2)}}_{p_{i-k+2}}\right\}.$$

The set $\Omega(k)$ is decomposed into two partitions. The first one is $\Omega^e := \{u^{(j)} \in \Omega(k) : j \in \mathbb{E}\}$, which is the set of all derivatives $u^{(j)}$ such that the integers $j \in \mathbb{E}$, where $\mathbb{E}$ is the set of even integers without zero. The second partition is $\Omega^o := \{u^{(j)} \in \Omega(k) : j \in \mathbb{O}\}$ for which $j \in \mathbb{O}$, where $\mathbb{O}$ is the set of odd integers without zero. The cardinals of Ω^e and Ω^o are denoted by $|\Omega^e|$ and $|\Omega^o|$, respectively. We have by definition of Bell's polynomial:

$$p_1+p_2+\cdots = k = |\Omega^e| + |\Omega^o| \tag{A3}$$

$$\sum_{\forall u^{(j)}\in\Omega(k)} j = i+1. \tag{A4}$$

Case 1: $i+1$ is even and k is even.
$\hookrightarrow$ Let us assume that $|\Omega^e|$ is even. As k is even this implies, from Eq.(A3), that $|\Omega^o|$ is even too. Since $u \in \Gamma^{\to s+1}$, we have the following implication:

$$\left[\prod_{\forall u^{(j)} \in \Omega^e}^{|\Omega^e|} u^{(j)} \geqslant 0 \ , \ \prod_{\forall u^{(j)} \in \Omega^o}^{|\Omega^o|} u^{(j)} \geqslant 0\right] \implies \left[B_{i+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(i-k+2)}\right) \geqslant 0\right].$$

$\hookrightarrow$ Assume that $|\Omega^e|$ is odd, then $|\Omega^o|$ is odd from Eq.(A3). Then, we deduce, from Eq.(A4), that:

$$\left[\left(\sum_{\forall u^{(j)} \in \Omega^o} j\right) \text{ is odd} \ , \ \left(\sum_{\forall u^{(j)} \in \Omega^e} j\right) \text{ is even}\right] \implies \left[\left(\sum_{\forall u^{(j)} \in \{\Omega^o \cup \Omega^e\}} j = i+1\right) \text{ is odd}\right].$$

Since the term $i+1$ has been assumed to be even, then it yields a contradiction: $|\Omega^e|$ and $|\Omega^o|$ cannot be odd when $i+1$ and k are even.

Case 2: $i+1$ is even and k is odd.
$\hookrightarrow$ Let us assume that $|\Omega^e|$ is odd. As k is odd, we get from Eq.(A3) that $|\Omega^o|$ is even. Since $u \in \Gamma^{\to s+1}$, we have:

$$\left[\prod_{\forall u^{(j)} \in \Omega^e}^{|\Omega^e|} u^{(j)} \leqslant 0 \ , \ \prod_{\forall u^{(j)} \in \Omega^o}^{|\Omega^o|} u^{(j)} \geqslant 0\right] \implies \left[B_{i+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(i-k+2)}\right) \leqslant 0\right].$$

$\hookrightarrow$ If $|\Omega^e|$ is even, then $|\Omega^o|$ is odd by Eq.(A3). This case is impossible since $i+1$ is even whereas we obtain a contradiction:

$$\left[\left(\sum_{\forall u^{(j)} \in \Omega^o} j\right) \text{ is odd} \ , \ \left(\sum_{\forall u^{(j)} \in \Omega^e} j\right) \text{ is even}\right] \implies \left[\left(\sum_{\forall u^{(j)} \in \{\Omega^o \cup \Omega^e\}} j = i+1\right) \text{ is odd}\right].$$

Case 3: $i+1$ is odd and k is even.
$\hookrightarrow$ Let us assume that $|\Omega^e|$ is odd, then $|\Omega^o|$ is odd by Eq.(A3). Since $u \in \Gamma^{\to s+1}$, we have:

$$\left[\prod_{\forall u^{(j)} \in \Omega^e}^{|\Omega^e|} u^{(j)} \leqslant 0 \ , \ \prod_{\forall u^{(j)} \in \Omega^o}^{|\Omega^o|} u^{(j)} \geqslant 0\right] \implies \left[B_{i+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(i-k+2)}\right) \leqslant 0\right].$$

$\hookrightarrow$ If $|\Omega^e|$ is even, then $|\Omega^o|$ is even by Eq.(A3). We have a contradiction since $i+1$ is assumed to be odd whereas:

$$\left[\left(\sum_{\forall u^{(j)} \in \Omega^o} j\right) \text{ is even} \ , \ \left(\sum_{\forall u^{(j)} \in \Omega^e} j\right) \text{ is even}\right] \implies \left[\left(\sum_{\forall u^{(j)} \in \{\Omega^o \cup \Omega^e\}} j = i+1\right) \text{ is even}\right].$$

Case 4: $i+1$ is odd and k is odd.
$\hookrightarrow$ Let us assume that $|\Omega^e|$ is even, thus $|\Omega^o|$ is odd by Eq.(A3). Since $u \in \Gamma^{\to s+1}$, we have:

$$\left[\prod_{\forall u^{(j)} \in \Omega^e}^{|\Omega^e|} u^{(j)} \geqslant 0 \ , \ \prod_{\forall u^{(j)} \in \Omega^o}^{|\Omega^o|} u^{(j)} \geqslant 0\right] \implies \left[B_{i+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(i-k+2)}\right) \geqslant 0\right].$$

$\hookrightarrow$ If $|\Omega^e|$ is odd, then $|\Omega^o|$ is even by Eq.(A3). This yields a contradiction since $i+1$ is assumed to be odd whereas:

$$\left[\left(\sum_{\forall u^{(j)} \in \Omega^o} j\right) \text{ is even} \ , \ \left(\sum_{\forall u^{(j)} \in \Omega^e} j\right) \text{ is even}\right] \implies \left[\left(\sum_{\forall u^{(j)} \in \{\Omega^o \cup \Omega^e\}} j = i+1\right) \text{ is even}\right].$$

Final Remark: By definition, we have $B_{i+1,1}(\cdot) = [u^{(i+1)}]^1$ and $B_{i+1,i+1}(\cdot) = [u^{(1)}]^{i+1}$. Since $g \in \Gamma^{\to s}$, we have therefore $(-1)^k g^{(k)} \circ u \leqslant 0$, for all $k = 1, \ldots, i$. Then, from Cases 1 to 4 that:

$$(-1)^{i+1}(g^{(k)} \circ u) \cdot B_{i+1,k}\left(u^{(1)}, u^{(2)}, \ldots, u^{(i-k+2)}\right) \leqslant 0, \ \forall k = 1, \ldots, i.$$

Hence, from Eq.(A2), $(-1)^{s+1} g^{\star(s+1)} \geqslant 0$, which ends the proof.

$\square$